IA Analysis I

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Abstract

These notes are adapted from my notes taken in *Analysis I* from Part IA of the Mathematical Tripos at the University of Cambridge in 2006-2007, lectured by Prof. Andrew Thomason. Some parts are taken from my notes in *Numbers and Sets*, lectured by Prof. Imre Leader.

Z notation is used in many places; see my notes on *Sets, Logic, Relations, and Functions* for definitions of unfamiliar symbols and words.

1 Sequences and Series

1.1 Bounds

$\mathbb{R}$, like $\mathbb{Q}$, is an *ordered field*.[1]

Definition 1.1.1. *For $S : \mathbb{P}\,\mathbb{R}, m : \mathbb{R}$,*

1. *m is an* ***upper bound*** *of S if $\forall\, x : S \bullet x \leqslant m$.*
2. *m is a* ***lower bound*** *of S if $\forall\, x : S \bullet x \geqslant m$.*
3. *m is a* ***supremum*** *or "least upper bound" of S if m is an upper bound of S and a lower bound of the set of upper bounds of S.*[2]
4. *m is an* ***infimum*** *or "greatest lower bound" of S if m is a lower bound of S and an upper bound of the set of lower bounds of S.*[3]
5. *S is* ***bounded-above*** *in $\mathbb{R}$ if it has an upper bound in $\mathbb{R}$,* ***bounded-below*** *in $\mathbb{R}$ if it has a lower bound in $\mathbb{R}$, and* ***bounded*** *in $\mathbb{R}$ if it is both bounded-above and bounded-below in $\mathbb{R}$.*

For $f : S \to \mathbb{R}$, when we refer to bounds of f we mean bounds of $\operatorname{ran} f$.

Proposition 1.1.2. *For $S : \mathbb{P}\,\mathbb{R}, m, m' : \mathbb{R}$,*

1. *If m, m' are suprema of S, then $m = m'$.*
2. *If m, m' are infima of S, then $m = m'$.*
3. *m is a supremum of S iff $-m$ is an infimum of $\{\, x : S \bullet -x \,\}$.*

Proof.

1. $m' \geqslant m$ and $m \geqslant m'$.
2. $m' \leqslant m$ and $m \leqslant m'$.
3. $\forall\, x : S \bullet (m \geqslant x) \Leftrightarrow (-m \leqslant -x)$.

□

[1] I.e. we can add, subtract, multiply and divide according to the usual rules, and for x, y, exactly one of $x < y$, $x = y$ or $x > y$ is true.

[2] I.e. $\forall\, x : S \bullet x \leqslant m$, and $\forall\, b : \mathbb{R} \mid (\forall\, x : S \bullet b \leqslant x) \bullet b \geqslant m$.

[3] I.e. $\forall\, x : S \bullet x \geqslant m$, and $\forall\, b : \mathbb{R} \mid (\forall\, x : S \bullet b \geqslant x) \bullet b \leqslant m$.

Axiom 1.1.3. *Every non-empty bounded-above subset $S : \mathbb{P}\mathbb{R}$ has a supremum* $\sup S$*.*

By Proposition 1.1.2, every non-empty bounded-below subset $S : \mathbb{P}\mathbb{R}$ has an infimum $\inf S$, and when they are defined, $\sup S$ and $\inf S$ are each unique.

Theorem 1.1.4. $\forall\, x : \mathbb{R}, n : \mathbb{N}_+ \mid x \geqslant 0 \bullet \exists! r : \mathbb{R} \mid r \geqslant 0 \bullet r^n = x.$

Proof. Let $r = \sup\{\, y : \mathbb{R} \mid y^n \leqslant x \,\}$,[4] $\epsilon = |r^n - x|$, and $\delta = \min\left\{\frac{\epsilon}{(1+r)^n}, 1\right\}$.

If $r^n < x$, then

$$(r+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} r^k \delta^{n-k} \leqslant r^n + \sum_{k=0}^{n-1} \binom{n}{k} r^k \delta \leqslant r^n + \delta(1+r)^n \leqslant r^n + \epsilon = x$$

and so $r + \delta > r$ is in the set, a contradiction.

Otherwise, if $r^n > x$, then

$$(r-\delta)^n = \sum_{k=0}^{n} \binom{n}{k} r^k (-\delta)^{n-k} \geqslant r^n - \sum_{k=0}^{n-1} \binom{n}{k} r^k \delta \geqslant r^n - \delta(1+r)^n \geqslant r^n - \epsilon = x$$

and so $r - \delta < r$ is an upper bound, a contradiction.

Therefore, $r^n = x$.[5]

Uniqueness is trivial. □

I.e. every non-negative x has an nth root $\sqrt[n]{x}$.

Definition 1.1.5. *For $x : \mathbb{R}, \frac{p}{q} : \mathbb{Q} \mid x \geqslant 0$, $x^{\frac{p}{q}} = \sqrt[q]{x}^p$.*[6]

Theorem 1.1.6 (Axiom of Archimedes)**.** $\forall\, r : \mathbb{R} \bullet \exists\, n : \mathbb{N} \bullet n > r.$

I.e. $\mathbb{N}$ is not bounded-above in $\mathbb{R}$.[7]

Proof. Suppose $\mathbb{N}$ is bounded-above in $\mathbb{R}$. $\mathbb{N} \neq \emptyset$, so let $M = \sup \mathbb{N}$. $M - 1$ is not an upper bound of $\mathbb{N}$, so $\exists\, n : \mathbb{N} \bullet n > M - 1$. Hence $\mathbb{N} \ni n + 1 > M$, so M is not an upper bound of $\mathbb{N}$, a contradiction. □

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[4] The set is non-empty as it contains 0, and is bounded-above by $\max\{1, x\}$.

[5] This is an example of an "ϵ/δ proof"; given an arbitrarily small change ϵ in the output, we look for a change δ in the input which is sufficiently small to produce it.

[6] We will take "$\frac{p}{q} : \mathbb{Q}$" to mean $p, q : \mathbb{Z} \mid q > 0$ with p, q coprime. In fact, Definition 1.1.5 is well-defined whether or not p, q are coprime.

[7] So e.g. ∞ is not a real number.

Corollary 1.1.7. $\inf\{ n : \mathbb{N}_+ \bullet \frac{1}{n} \} = 0$.

Proof. The set is bounded-below by 0, so let $\epsilon \geqslant 0$ be the infimum. We have $\forall\, n : \mathbb{N}_+ \bullet \epsilon \leqslant \frac{1}{n}$. If $\epsilon > 0$, then $\forall\, n : \mathbb{N}_+ \bullet \frac{1}{\epsilon} \geqslant n$, so $\frac{1}{\epsilon}$ is an upper bound of $\mathbb{N}$, contradicting Theorem 1.1.6. Therefore, $\epsilon = 0$. □

1.2 Sequences and Convergence

Definition 1.2.1.

1. *For a set X, a "**sequence** of X" is a function $: \mathbb{N} \to X$.*

2. *A **real sequence** is a sequence of $\mathbb{R}$.*

3. *A **complex sequence** is sequence of $\mathbb{C}$.*

4. *If a, b are sequences, then b is a **subsequence** of a if, for some $N : \mathbb{P}\,\mathbb{N}$ enumerated[8] in order as $n_0 < n_1 < \ldots$, $\forall\, i : \mathbb{N} \bullet b(i) = a(n_i)$.*

For a sequence a, we will write a_k to mean $a(k)$, and $(a_n)_{n:\mathbb{N}}$ to mean a. Where it is clearer, we will write "sequence" to mean "real sequence or complex sequence".

Definition 1.2.2. *For a sequence $(a_n)_{n:\mathbb{N}}$,*

1. *For $l : \mathbb{R}$ or $\mathbb{C}$, "$a_n \longrightarrow l$ as $n \longrightarrow \infty$", or "l is a **limit** of $(a_n)_{n:\mathbb{N}}$", if $\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |a_n - l| < \epsilon$.*

   I.e. for any distance ϵ, however small, the terms in the sequence eventually get, and stay, within ϵ of l.

2. *$(a_n)_{n:\mathbb{N}}$ is **convergent** if $\exists\, l : \mathbb{R}$ or $\mathbb{C} \bullet a_n \longrightarrow l$ as $n \longrightarrow \infty$.*

For a real sequence $(a_n)_{n:\mathbb{N}}$,

3. *"$a_n \longrightarrow \infty$ as $n \longrightarrow \infty$" if $\forall\, x : \mathbb{R} \bullet \exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n > x$.*

   I.e. every $x : \mathbb{R}$ eventually becomes a lower bound of the rest of the sequence.

4. *"$a_n \longrightarrow -\infty$ as $n \longrightarrow \infty$" if $-a_n \longrightarrow \infty$ as $n \longrightarrow \infty$.*

   I.e. every $x : \mathbb{R}$ eventually becomes an upper bound of the rest of the sequence.

5. *$(a_n)_{n:\mathbb{N}}$ is **divergent** if it is not convergent.*

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[8] I.e. $(n_i)_{i:\mathbb{N}}$ is a strictly-increasing sequence of $\mathbb{N}$.

We will typically write "$a_n \longrightarrow l$" instead of "$a_n \longrightarrow l$ as $n \longrightarrow \infty$". Also, we will often use the letter a instead of l for a limit.[9]

We say a_n "tends to" a, or a_n "converges to" a, a_n "converges", or a_n "diverges". Where $a_n \longrightarrow \pm\infty$ we say a_n "diverges to" $\pm\infty$.

A complex sequence $(z_n)_{n:\mathbb{N}}$ diverges to ∞ if $(|z_n|)_{n:\mathbb{N}}$ diverges to ∞.

Note that while a real sequence cannot both converge and diverge to $\pm\infty$, it is not necessarily the case that a sequence either converges or diverges to $\pm\infty$.

Proposition 1.2.3. *If a complex sequence $(z_n)_{n:\mathbb{N}}$ is convergent, then $(\mathrm{Re}(z_n))_{n:\mathbb{N}}$ and $(\mathrm{Im}(z_n))_{n:\mathbb{N}}$ are convergent.*[10]

Proof. Suppose $z_n \longrightarrow z$ for some $z : \mathbb{C}$. Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |z_n - z| < \epsilon$, so $|\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \epsilon$. Hence, $\mathrm{Re}(z_n) \longrightarrow \mathrm{Re}(z)$.

Similarly for $(\mathrm{Im}(z_n))_{n:\mathbb{N}}$. □

Proposition 1.2.4. *For sequences $(a_n)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}$ and $a, b : \mathbb{R}$ or $\mathbb{C}$,*

1. *If $a_n \longrightarrow a$ and $a_n \longrightarrow b$, then $a = b$.*
2. *If $a_n = b_n$ for all but finitely many $n : \mathbb{N}$, and $a_n \longrightarrow a$, then $b_n \longrightarrow a$.*
3. *If $\forall\, n : \mathbb{N} \bullet a_n = a$, then $a_n \longrightarrow a$.*
4. *If $(b_n)_{n:\mathbb{N}}$ is a subsequence of $(a_n)_{n:\mathbb{N}}$, and $a_n \longrightarrow a$, then $b_n \longrightarrow a$.*
5. *If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $\forall\, \lambda, \mu : \mathbb{R}$ or $\mathbb{C} \bullet$ $(\lambda\, a_n + \mu\, b_n) \longrightarrow (\lambda\, a + \mu\, b)$.*[11]
6. *If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $a_n b_n \longrightarrow ab$.*
7. *If $a_n \longrightarrow a \neq 0$ and $\forall\, n : \mathbb{N} \bullet a_n \neq 0$, then $\dfrac{1}{a_n} \longrightarrow \dfrac{1}{a}$.*

Proof.

1. For real sequences, suppose $a \neq b$; wlog[12] $a < b$.[13]

[9] This is an abuse of notation, as we defined a sequence a_n to be a function a, so we are using the same letter to mean two different things. However, for notational convenience, we will forget about the function a, and refer to the sequence as $(a_n)_{n:\mathbb{N}}$.

[10] The converse follows immediately from Proposition 1.2.4.5.

[11] I.e. $\lim_{n\to\infty}$ is a **linear operator** on convergent sequences.

[12] "Without loss of generality", i.e. the extra assumption doesn't weaken the result.

[13] I.e. otherwise we can simply relabel a, b.

By taking $\epsilon = \frac{b-a}{2}$, $\exists\, M, N : \mathbb{N}$ with $\forall\, n : \mathbb{N} \mid n \geqslant M \bullet a_n < a + \epsilon$ and $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n > b - \epsilon$. Then $a + \epsilon = b - \epsilon < a_{\max\{M,N\}} < a + \epsilon$, a contradiction.

For complex sequences, by Proposition 1.2.3, $\mathrm{Re}(a) = \mathrm{Re}(b)$, and $\mathrm{Im}(a) = \mathrm{Im}(b)$, so $a = b$.

2. Let $M = \max\{\, n : \mathbb{N} \mid a_n \neq b_n \,\}$.[14]

   Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |a_n - a| < \epsilon$.

   Hence, $\forall\, n : \mathbb{N} \mid n \geqslant \max M, N \bullet |b_n - a| = |a_n - a| < \epsilon$.

3. $\forall\, \epsilon : \mathbb{R}_+, n : \mathbb{N} \bullet |a_n - a| = 0 < \epsilon$.

4. Let $m : \mathbb{N} \rightarrowtail \mathbb{N}$ be the enumeration of $(b_n)_{n:\mathbb{N}}$ in $(a_n)_{n:\mathbb{N}}$.[15] Note that $\forall\, n : \mathbb{N} \bullet m(n) \geqslant n$.[16]

   Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |b_n - a| = \left|a_{m(n)} - a\right| < \epsilon$.

5. Given $\epsilon > 0$, let $\epsilon_\lambda = \frac{\epsilon}{2\max\{1,|\lambda|\}}$, $\epsilon_\mu = \frac{\epsilon}{2\max\{1,|\mu|\}}$.

   $\exists\, M : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant M \bullet |a_n - a| < \epsilon_\lambda$, and
   $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |b_n - b| < \epsilon_\mu$.

   Then by the triangle inequality,[17] $\forall\, n : \mathbb{N} \mid n \geqslant \max\{M, N\}$,
   $|(\lambda\, a_n + \mu\, b_n) - (\lambda\, a + \mu\, b)| \leqslant |\lambda|\, |a_n - a| + |\mu|\, |b_n - b| < |\lambda|\, \epsilon_\lambda + |\mu|\, \epsilon_\mu \leqslant \epsilon$.

6. *Claim:* If $a = b = 0$, then $a_n b_n \longrightarrow 0$.

   Proof of claim: $\exists\, M : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant M \bullet |a_n| < 1$.

   Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |b_n| < \epsilon$.

   Hence $\forall\, n : \mathbb{N} \mid n \geqslant \max\{M, N\} \bullet |a_n b_n| = |a_n| \cdot |b_n| < |b_n| < \epsilon$.

   By (5), $(a_n - a) \longrightarrow 0$ and $(b_n - b) \longrightarrow 0$.

   By the claim, $(a_n - a)(b_n - b) \longrightarrow 0$.

   By (3), $ab \longrightarrow ab$, and by (5), $ab_n \longrightarrow ab$ and $ba_n \longrightarrow ab$.

   Therefore by (5), $a_n b_n = ((a_n - a)(b_n - b) + ab_n + ba_n - ab) \longrightarrow ab$.

[14] Every finite subset of $\mathbb{N}$ has a greatest element.

[15] I.e. the nth element of the subsequence is the $m(n)$th element of the sequence.

[16] E.g. let $S = \{\, k : \mathbb{N} \mid k \leqslant n \bullet m(k) \,\}$, then $m(n) = \max S$ as m is strictly-increasing, $\max S \geqslant \#S$ as $S \subseteq \{0, 1, \ldots \max S\}$, and $\#S = n$ as m is injective.

[17] $\forall\, x, y : \mathbb{C} \bullet |x + y| \leqslant |x| + |y|$.

7. Given $\epsilon > 0$, wlog $\epsilon < \frac{1}{|a|}$.[18] Note that $0 < \frac{|a|^2\epsilon}{1+|a|\epsilon} < |a|$.

   $\forall\, x : \mathbb{R}^* \text{ or } \mathbb{C}^*$, if $|x - a| < \frac{|a|^2\epsilon}{1+|a|\epsilon} < |a|$, then by the reverse triangle inequality,[19] $|x| \geqslant |a| - |x-a| > |a| - \frac{|a|^2\epsilon}{1+|a|\epsilon} = \frac{|a|}{1+|a|\epsilon} > 0$,
   so $\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|a-x|}{|ax|} < \frac{|a|^2\epsilon}{1+|a|\epsilon} \cdot \frac{1}{|a||x|} = \frac{|a|\epsilon}{1+|a|\epsilon} \cdot \frac{1}{|x|} < \frac{|a|\epsilon}{1+|a|\epsilon} \cdot \frac{1+|a|\epsilon}{|a|} = \epsilon$.

   Hence, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N$, $|a_n - a| < \frac{|a|^2\epsilon}{1+|a|\epsilon}$, so $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \epsilon$.

□

Definition 1.2.5. *If* $a_n \longrightarrow a$, *then* $\lim\limits_{n\to\infty} a_n = a$ *is* the *limit of* $(a_n)_{n:\mathbb{N}}$.

By Proposition 1.2.4.1, this is well-defined.

Lemma 1.2.6. *If* $(b_{1,n})_{n:\mathbb{N}}, \ldots (b_{k,n})_{n:\mathbb{N}}$ *are a partition of* $(a_n)_{n:\mathbb{N}}$ *for some* $k : \mathbb{N}$, *and* $\exists\, l : \mathbb{R}$ *or* $\mathbb{C} \bullet \forall\, i : \mathbb{N} \mid 1 \leqslant i \leqslant k \bullet b_{i,n} \longrightarrow l$, *then* $a_n \longrightarrow l$.[20]

Proof. Given $\epsilon > 0$, $\forall\, i : \mathbb{N} \mid 1 \leqslant i \leqslant k \bullet \exists\, N_i : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N_i \bullet |b_{i,n} - l| < \epsilon$. Let $N = \max_{i=1}^{k} N_i$, then $\forall\, n : \mathbb{N} \mid n \geqslant N$, $a_n = b_{i,m_i(n)}$ for some $1 \leqslant i \leqslant k$, and $m_i(n) \geqslant n \geqslant N \geqslant N_i$, so $|a_n - l| < \epsilon$. □

1.3 Convergence and Bounds

Lemma 1.3.1. *Every convergent sequence is bounded.*[21]

Proof. If $(a_n)_{n:\mathbb{N}}$ is a real sequence converging to $a : \mathbb{R}$, then $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |a_n - a| < 1$.[22] Let $L = \min\{a_0, \ldots a_{N-1}, a - 1\}$ and $U = \max\{a_0, \ldots a_{N-1}, a + 1\}$. Then $\forall\, n : \mathbb{N}$, whether $n < N$ or $n \geqslant N$, $L \leqslant a_n \leqslant U$, and so L and U are lower and upper bounds for $(a_n)_{n:\mathbb{N}}$ respectively.

If $(z_n)_{n:\mathbb{N}}$ is a complex sequence converging to $z : \mathbb{C}$, then by Proposition 1.2.3, $\mathrm{Re}(z_n) \longrightarrow \mathrm{Re}(z)$ and $\mathrm{Im}(z_n) \longrightarrow \mathrm{Im}(z)$, so $(\mathrm{Re}(z_n))_{n:\mathbb{N}}$ and $(\mathrm{Im}(z_n))_{n:\mathbb{N}}$ are bounded, hence $(|z_n|)_{n:\mathbb{N}}$ is bounded. □

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[18] The result for e.g. $\epsilon = \frac{1}{2|a|}$ implies the result for $\epsilon > \frac{1}{2|a|}$.

[19] $\forall\, x, y : \mathbb{C} \bullet ||x| - |y|| \leqslant |x - y|$.

[20] This is a partial converse to Proposition 1.2.4.4.

[21] I.e. if $(a_n)_{n:\mathbb{N}}$ is convergent, then $\operatorname{ran} a$ is bounded. For complex sequences, we mean $(|a_n|)_{n:\mathbb{N}}$ is bounded.

[22] I.e. take N for $\epsilon = 1$.

Theorem 1.3.2. *If $(a_n)_{n:\mathbb{N}}$ and $(b_n)_{n:\mathbb{N}}$ are real sequences converging to a and b respectively, and $\forall\, n : \mathbb{N} \bullet a_n \leqslant b_n$, then $a \leqslant b$.*

Proof. Suppose $a > b$. Let $\epsilon = \frac{a-b}{2}$.

- $a_n \longrightarrow a$ so $\exists\, M : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant M \bullet |a_n - a| < \epsilon$.
- $b_n \longrightarrow b$ so $\exists\, M' : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant M' \bullet |b_n - b| < \epsilon$.

Let $N = \max\{M, M'\}$. Then

$$|a - b| = |(b_N - b) - (a_N - a)| \leqslant |b_N - b| + |a_N - a| < \epsilon + \epsilon = a - b$$

is a contradiction. □

Corollary 1.3.3. *If $(a_n)_{n:\mathbb{N}}$ is a real sequence converging to a, then* $\inf\{\, n : \mathbb{N} \bullet a_n \,\} \leqslant a \leqslant \sup\{\, n : \mathbb{N} \bullet a_n \,\}$.

Proof. Let $s = \sup\{\, n : \mathbb{N} \bullet a_n \,\}$.[23] $\forall\, n : \mathbb{N} \bullet a_n \leqslant s$, and by Proposition 1.2.4.3, $s \longrightarrow s$. Hence, by Theorem 1.3.2, $a \leqslant s$.

Similarly for the infimum. □

More generally, if m is a bound of $(a_n)_{n:\mathbb{N}}$, then m is also a bound of $\lim_{n\to\infty} a_n$.

Theorem 1.3.4. *For real sequences $(a_n)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}, (c_n)_{n:\mathbb{N}}$, if $a_n \longrightarrow l$ and $c_n \longrightarrow l$ for some $l : \mathbb{R}$, and $\forall\, n : \mathbb{N} \bullet a_n \leqslant b_n \leqslant c_n$, then $b_n \longrightarrow l$.*

Proof. By Proposition 1.2.4.5, $(c_n - a_n) \longrightarrow 0$. Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |b_n - a_n| \leqslant |c_n - a_n| < \epsilon$, hence $(b_n - a_n) \longrightarrow 0$. Then $b_n = ((b_n - a_n) + a_n) \longrightarrow 0 + l = l$. □

Corollary 1.3.5.

1. *$|a_n| \longrightarrow 0$ iff $a_n \longrightarrow 0$.*
2. *If $a_n \longrightarrow 0$ and $(b_n)_{n:\mathbb{N}}$ is bounded, then $a_n b_n \longrightarrow 0$.*

Proof.

1. $||a_n|| < \epsilon$ iff $|a_n| < \epsilon$.
2. By assumption, $\exists\, B : \mathbb{R} \bullet \forall\, n : \mathbb{N} \bullet |b_n| \leqslant B$. By (1), $B\,|a_n| \longrightarrow 0$ and $-B\,|a_n| \longrightarrow 0$.

   $\forall\, n : \mathbb{N} \bullet -B\,|a_n| \leqslant |a_n b_n| \leqslant B\,|a_n|$. Therefore, by Theorem 1.3.4 $|a_n b_n| \longrightarrow 0$, and by (1), $a_n b_n \longrightarrow 0$.

□

[23] The supremum exists by Lemma 1.3.1.

Proposition 1.3.6. *If $S : \mathbb{P}\,\mathbb{R}$ is non-empty and bounded-above, then there is a sequence $(a_n)_{n:\mathbb{N}}$ converging to $\sup S$, with $\forall\, n : \mathbb{N} \bullet a_n \in S$.*

Proof. $\forall\, n : \mathbb{N} \bullet \sup S - \frac{1}{n}$ is not an upper bound of S, so $\exists\, a_n : S \bullet a_n > \sup S - \frac{1}{n}$. Given $\epsilon > 0$, by Theorem 1.1.6 $\exists\, N : \mathbb{N} \bullet N > \frac{1}{\epsilon}$, so $\forall\, n : \mathbb{N} \bullet |a_n - \sup S| < \frac{1}{n} \leqslant \frac{1}{N} < \epsilon$. Hence, $a_n \longrightarrow \sup S$. □

In particular, $\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, x : S \bullet x > \sup S - \epsilon$.

Similarly, if S is non-empty and bounded-below, then there is a sequence $(a_n)_{n:\mathbb{N}}$ converging to $\inf S$, with $\forall\, n : \mathbb{N} \bullet a_n \in S$. In particular, $\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, x : S \bullet x < \inf S + \epsilon$.

1.4 Monotonic Sequences

Definition 1.4.1. *Let $(a_n)_{n:\mathbb{N}}$ be a real sequence.*

1. *$(a_n)_{n:\mathbb{N}}$ is **non-decreasing** if $\forall\, m, n : \mathbb{N} \mid m < n \bullet a_m \leqslant a_n$.*
2. *$(a_n)_{n:\mathbb{N}}$ is **non-increasing** if $\forall\, m, n : \mathbb{N} \mid m < n \bullet a_m \geqslant a_n$.*
3. *$(a_n)_{n:\mathbb{N}}$ is **monotonic** if it is non-decreasing or non-increasing.*
4. *$(a_n)_{n:\mathbb{N}}$ is **strictly-increasing** if $\forall\, m, n : \mathbb{N} \mid m < n \bullet a_m < a_n$.*
5. *$(a_n)_{n:\mathbb{N}}$ is **strictly-decreasing** if $\forall\, m, n : \mathbb{N} \mid m < n \bullet a_m > a_n$.*
6. *$(a_n)_{n:\mathbb{N}}$ is **strictly-monotonic** if it is strictly-increasing or strictly-decreasing.*

Theorem 1.4.2.

1. *Every bounded monotonic[24] sequence is convergent.*
2. *Every unbounded monotonic[25] sequence diverges to $\pm\infty$.*

Proof. Wlog $(a_n)_{n:\mathbb{N}}$ is non-decreasing.[26]

1. Let $a = \sup_{n:\mathbb{N}} a_n$.

   Given $\epsilon > 0$, $a - \epsilon < a$ so $a - \epsilon$ is not an upper bound of $(a_n)_{n:\mathbb{N}}$. Hence, $\exists\, N : \mathbb{N} \bullet a_N > a - \epsilon$.

   $(a_n)_{n:\mathbb{N}}$ is non-decreasing, so $\forall\, n : \mathbb{N} \mid n \geqslant N$, $a - \epsilon < a_N \leqslant a_n$ and $a_n \leqslant a < a + \epsilon$, so $|a_n - a| < \epsilon$. Therefore, $a_n \longrightarrow a$.

[24] I.e. bounded-above non-decreasing, or bounded-below non-increasing.

[25] I.e. unbounded-above non-decreasing, or unbounded-below non-increasing.

[26] I.e. for $(a_n)_{n:\mathbb{N}}$ non-increasing, $(-a_n)_{n:\mathbb{N}}$ is non-decreasing, $-a_n \longrightarrow -a$ iff $a_n \longrightarrow a$, and $-a_n \longrightarrow -\infty$ iff $a_n \longrightarrow \infty$.

2. $(a_n)_{n:\mathbb{N}}$ is unbounded, so $\forall\, x : \mathbb{R} \bullet \exists\, N : \mathbb{N} \bullet a_N > x$. $(a_n)_{n:\mathbb{N}}$ is non-decreasing, so $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n \geqslant a_N > x$, hence $a_n \longrightarrow \infty$.

□

I.e. every monotonic sequence either converges or diverges to $\pm\infty$.

Lemma 1.4.3. *Every unbounded sequence has a subsequence diverging to* $\pm\infty$*.*

Proof. Suppose $(a_n)_{n:\mathbb{N}}$ is unbounded-above.[27] Let $m_0 = 0$, and where m_n is defined, let $M_n = \max\{a_0, \dots a_{m_n}, n\}$. M_n is not an upper bound of $(a_n)_{n:\mathbb{N}}$, so $\exists\, m_{n+1} : \mathbb{N} \bullet a_{m_{n+1}} > M_n$. Then $a_{m_{n+1}} > a_{m_n}$, and $m_{n+1} > m_n$.[28] Inductively, $(a_{m_n})_{n:\mathbb{N}}$ is a subsequence. $\forall\, n : \mathbb{N} \bullet a_{m_n} > n$, so $a_{m_n} \longrightarrow \infty$. □

1.5 Cauchy Sequences

Definition 1.5.1. *For a real sequence* $(a_n)_{n:\mathbb{N}}$*,*

1. $\liminf a_n = \lim_{n\to\infty} \inf\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$.

2. $\limsup a_n = \lim_{n\to\infty} \sup\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$.

Proposition 1.5.2.

1. *If* $(a_n)_{n:\mathbb{N}}$ *is bounded-above, then* $\liminf a_n$ *exists.*

2. *If* $(a_n)_{n:\mathbb{N}}$ *is bounded-below, then* $\limsup a_n$ *exists.*

Proof. Let $i_n = \inf\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$. $(i_n)_{n:\mathbb{N}}$ is non-decreasing, as in general if $A \subseteq B$, then $\inf A \geqslant \inf B$. Also, note that $m : \mathbb{R}$ is an upper bound of $(a_n)_{n:\mathbb{N}}$ iff it is an upper bound of $(i_n)_{n:\mathbb{N}}$. Therefore, if $(a_n)_{n:\mathbb{N}}$ is bounded-above, then by Theorem 1.4.2.1, $(i_n)_{n:\mathbb{N}}$ is convergent.

Similarly for $\limsup a_n$. □

By Theorem 1.3.2, where they exist, $\liminf a_n \leqslant \lim\limits_{n\to\infty} a_n \leqslant \limsup a_n$.

Lemma 1.5.3. *For* $\epsilon > 0$*, if* $\liminf a_n = I$*,*

1. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n > I - \epsilon$.

2. $\forall\, N : \mathbb{N} \bullet \exists\, n : \mathbb{N} \mid n \geqslant N \bullet a_n < I + \epsilon$.

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[27] $(a_n)_{n:\mathbb{N}}$ unbounded-below is similar. For an unbounded complex sequence, $(|a_n|)_{n:\mathbb{N}}$ is similar.

[28] As $\forall\, k : \mathbb{N} \mid k \leqslant m_n \bullet a_k < a_{m_{n+1}}$.

If $\limsup a_n = S$*, then*

3. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n < S + \epsilon.$

4. $\forall\, N : \mathbb{N} \bullet \exists\, n : \mathbb{N} \mid n \geqslant N \bullet a_n > S - \epsilon.$

Proof. Let $I_n = \inf\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$.

1. Since $I_n \longrightarrow I$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n \geqslant I_n > I - \epsilon$.

2. Suppose not. Then $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n \geqslant I + \epsilon$. $(I_n)_{n:\mathbb{N}}$ is non-decreasing, so $\lim_{n\to\infty} I_n \geqslant I_N \geqslant I + \epsilon > I$, a contradiction.

Similarly for $\limsup a_n$. □

Theorem 1.5.4 (General Principle of Convergence)**.** *A real sequence* $(a_n)_{n:\mathbb{N}}$ *is convergent iff* $\liminf a_n = \limsup a_n$*.*

Proof. Let $I_n = \inf\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$ and $S_n = \sup\{\, k : \mathbb{N} \mid k \geqslant n \bullet a_k \,\}$.

Suppose $a_n \longrightarrow a$. Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N$, $|a_n - a| < \frac{\epsilon}{2}$, so by Lemma 1.5.3 $\forall\, n \geqslant N$,

- $a - \epsilon < a - \frac{\epsilon}{2} \leqslant I_n \leqslant a < a + \epsilon$, so $|I_n - a| < \epsilon$.
- $a - \epsilon < a \leqslant S_n \leqslant a + \frac{\epsilon}{2} < a + \epsilon$, so $|S_n - a| < \epsilon$.

Therefore, $\liminf a_n = a$ and $\limsup a_n = a$.

Conversely, $\forall\, n : \mathbb{N} \bullet I_n \leqslant a_n \leqslant S_n$, so if $I_n \longrightarrow a$ and $S_n \longrightarrow a$, then by Theorem 1.3.4, $a_n \longrightarrow a$. □

Definition 1.5.5. $(a_n)_{n:\mathbb{N}}$ *is* ***Cauchy*** *if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, N : \mathbb{N} \bullet \forall\, m, n : \mathbb{N} \mid m, n \geqslant N \bullet |a_n - a_m| < \epsilon$$

I.e. for any distance ϵ*, however small, the terms in the sequence eventually get, and stay, within* ϵ *of each other.*

Theorem 1.5.6. *A sequence is convergent iff it is Cauchy.*[29]

[29] I.e. $\mathbb{R}$ and $\mathbb{C}$ are **complete**.

Proof. Let $(a_n)_{n:\mathbb{N}}$ be a real sequence.

Suppose $a_n \longrightarrow a$. Given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet |a_n - a| < \frac{\epsilon}{2}$. Then $\forall\, m, n : \mathbb{N} \mid m, n \geqslant N$,

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leqslant |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so $(a_n)_{n:\mathbb{N}}$ is Cauchy.

Conversely, suppose $(a_n)_{n:\mathbb{N}}$ is Cauchy. $\exists\, N : \mathbb{N} \bullet \forall\, m, n : \mathbb{N} \mid m, n \geqslant N \bullet |a_n - a_m| < 1$,[30] so $\min\{a_0, \ldots a_{N-1}, a_N - 1\}$ and $\max\{a_0, \ldots a_{N-1}, a_N + 1\}$ are lower and upper bounds for $(a_n)_{n:\mathbb{N}}$ respectively. By Proposition 1.5.2, $\liminf a_n$ and $\limsup a_n$ both exist.

Let $I = \liminf a_n$ and $S = \limsup a_n$. Suppose $I \neq S$. Let $\epsilon = \frac{S-I}{3} > 0$.

- By Lemma 1.5.3.2, $\forall\, N : \mathbb{N} \bullet \exists\, m : \mathbb{N} \mid m \geqslant N \bullet a_m - I < \epsilon$.
- By Lemma 1.5.3.4, $\forall\, N : \mathbb{N} \bullet \exists\, n : \mathbb{N} \mid n \geqslant N \bullet S - a_n < \epsilon$.
- $(a_n)_{n:\mathbb{N}}$ is Cauchy, so $\exists\, N : \mathbb{N} \bullet \forall\, m, n : \mathbb{N} \mid m, n \geqslant B \bullet |a_n - a_m| < \epsilon$.

Hence, $|S - I| = |(S - a_n) + (a_n - a_m) + (a_m - I)| \leqslant |S - a_n| + |a_n - a_m| + |a_m - I| < \epsilon + \epsilon + \epsilon = S - I$, a contradiction.

Therefore, $I = S$ and by Theorem 1.5.4, $(a_n)_{n:\mathbb{N}}$ is convergent.

Hence, a complex sequence $(z_n)_{n:\mathbb{N}}$ converges iff $(\mathrm{Re}(z_n))_{n:\mathbb{N}}, (\mathrm{Im}(z_n))_{n:\mathbb{N}}$ converge, iff $(\mathrm{Re}(z_n))_{n:\mathbb{N}}, (\mathrm{Im}(z_n))_{n:\mathbb{N}}$ are Cauchy, iff $(z_n)_{n:\mathbb{N}}$ is Cauchy. □

Theorem 1.5.7 (Bolzano-Weierstrass Theorem)**.** *Every bounded real sequence has a Cauchy subsequence.*

Proof. Let $(a_n)_{n:\mathbb{N}}$ be a sequence, L_0 a lower bound, U_0 an upper bound, and $m_0 = 0$. Let $A(l, u) = \{\, k : \mathbb{N} \mid l \leqslant a_k \leqslant u \,\}$. Note that $A(L_0, U_0) = \mathbb{N}$ is infinite.

Where L_n, U_n, m_n are defined, let $C_n = \frac{L_n + U_n}{2}$. $A(L_n, U_n)$ is an infinite set, and $A(L_n, U_n) = A(L_n, C_n) \cup A(C_n, U_n)$.

If $A(L_n, C_n)$ is infinite, let $L_{n+1} = L_n$, $U_{n+1} = C_n$; otherwise $A(C_n, U_n)$ is infinite, so let $L_{n+1} = C_n$, $U_{n+1} = U_n$.[31]

Let $m_{n+1} = \min\{\, k : A(L_{n+1}, U_{n+1}) \mid k > m_n \,\}$.[32] By construction, $m_{n+1} > m_n$.

Inductively, $\forall\, n : \mathbb{N}$, m_n is defined. Let $b_n = a_{m_n}$.

[30] I.e. take N for $\epsilon = 1$.

[31] Concretely, we can keep halving the gap between the bounds, in such a way that infinitely many terms of the sequence remain between the bounds.

[32] This set is non-empty, as $A(L_{n+1}, U_{n+1})$ is not bounded-above by m_n.

For $p, q : \mathbb{N}$, $N = \min\{p, q\}$, $|b_q - b_p| \leqslant (U_N - L_N) = 2^{-N}(U_0 - L_0)$. Hence, given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet 2^{-N}(U_0 - L_0) < \epsilon$,[33] so $\forall\, p, q : \mathbb{N} \mid p, q \geqslant N \bullet |b_q - b_p| < \epsilon$. Hence, $(b_n)_{n:\mathbb{N}}$ is a Cauchy subsequence. □

In particular, every bounded real sequence has a convergent subsequence.

Corollary 1.5.8. *Every bounded complex sequence has a convergent subsequence.*

Proof. Let $(a_n)_{n:\mathbb{N}}$ be a bounded complex sequence.

$(\mathrm{Re}(a_n))_{n:\mathbb{N}}$ is bounded, so by Theorem 1.5.7, there is a subsequence $(b_n)_{n:\mathbb{N}}$ for which $(\mathrm{Re}(b_n))_{n:\mathbb{N}}$ converges.

$(\mathrm{Im}(b_n))_{n:\mathbb{N}}$ is bounded, so by Theorem 1.5.7, there is a subsequence $(c_n)_{n:\mathbb{N}}$ for which $(\mathrm{Im}(b_n))_{n:\mathbb{N}}$ converges.

By Proposition 1.2.4.4, $(\mathrm{Re}(c_n))_{n:\mathbb{N}}$ converges. Hence, $(c_n)_{n:\mathbb{N}}$ converges. □

1.6 Series

Definition 1.6.1. *For a sequence $(a_n)_{n:\mathbb{N}}$,*

1. *The sequence of **partial sums** $(S_n)_{n:\mathbb{N}}$ is given by $S_n = \sum_{k=0}^{n} a_k$.*[34]
2. *The **series** $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$, if it exists.*[35]
3. *$\sum_{n=0}^{\infty} a_n$ converges/diverges if $(S_n)_{n:\mathbb{N}}$ converges/diverges.*

If $(a_n)_{n:\mathbb{N}}$ is a real sequence then the series is a "real series", and if it is a complex sequence then the series is a "complex series".

Lemma 1.6.2. *If $\sum_{n=0}^{\infty} a_n$ exists, then $a_n \longrightarrow 0$.*

Proof. If $S_n \longrightarrow S$, then $a_n = (S_n - S_{n-1}) \longrightarrow (S - S) = 0$. □

The converse is false; there are divergent series whose terms tend to 0.

[33] E.g. as $2^{-n} \longrightarrow 0$.

[34] If a_n is defined only for $n \geqslant n_0$, then $S_n = \sum_{k=n_0}^{n} a_k$.

[35] Also, $\sum_{n=n_0}^{\infty} a_n = \lim_{n\to\infty}(S_n - S_{n_0})$.

Proposition 1.6.3. *For* $x : \mathbb{C}$, $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ *if* $|x| < 1$, *and the series diverges to* ∞ *if* $|x| > 1$.

Proof. $\forall\, n : \mathbb{N} \bullet (1-x) \displaystyle\sum_{k=0}^{n} x^n = 1 - x^{n+1}$, so for $x \neq 1$, $S_n = \dfrac{1-x^{n+1}}{1-x}$.

If $|x| < 1$, then $x^{n+1} \longrightarrow 0$, so by Proposition 1.2.4, $\dfrac{1-x^{n+1}}{1-x} \longrightarrow \dfrac{1}{1-x}$.

Otherwise, if $|x| > 1$ then $(x^{n+1})_{n:\mathbb{N}}$ diverges to ∞, so $\left(\dfrac{1-x^{n+1}}{1-x}\right)_{n:\mathbb{N}}$ diverges to ∞. □

Theorem 1.6.4 (General Principle of Convergence for Series)**.**

$$\sum_{n=0}^{\infty} a_n \textit{ converges iff } \forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, N : \mathbb{N} \bullet \forall\, m, n : \mathbb{N} \mid n \geqslant m \geqslant N \bullet \left|\sum_{k=m}^{n} a_k\right| < \epsilon.$$

Proof. By Theorem 1.5.6, $(S_n)_{n:\mathbb{N}}$ converges iff it is Cauchy, i.e. given $\epsilon > 0$,

$$\exists\, N : \mathbb{N} \bullet \forall\, m, n : \mathbb{N} \mid m, n' \geqslant N, \text{ wlog } n' \geqslant m,\ |S_{n'} - S_m| = \left|\sum_{k=m}^{n'-1} a_k\right| < \epsilon$$

If $n' = m$ then $|S_{n'} - S_m| = 0 < \epsilon$, otherwise let $n = n' + 1$, and the result follows. □

Proposition 1.6.5. *For series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$,

1. *If both series converge, then*

$$\forall\, \lambda, \mu : \mathbb{R} \textit{ or } \mathbb{C} \bullet \sum_{n=0}^{\infty} (\lambda\, a_n + \mu\, b_n) = \lambda \left[\sum_{n=0}^{\infty} a_n\right] + \mu \left[\sum_{n=0}^{\infty} b_n\right].$$ [36]

2. *If* $a_n = b_n$ *for all but finitely many* $n : \mathbb{N}$, *then* $\sum_{n=0}^{\infty} a_n$ *converges iff* $\sum_{n=0}^{\infty} b_n$ *converges.*

Proof. Let $(S_n)_{n:\mathbb{N}}, (T_n)_{n:\mathbb{N}}$ be the partial sums for $(a_n)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}$.

[36] I.e. $\sum_{n=0}^{\infty}$ is a linear operator on convergent series.

1. The partial sums for $(\lambda\, a_n + \mu\, b_n)_{n:\mathbb{N}}$ are $(\lambda\, S_n + \mu\, T_n)_{n:\mathbb{N}}$. If $S_n \longrightarrow S$ and $T_n \longrightarrow T$, then $(\lambda\, S_n + \mu\, T_n) \longrightarrow (\lambda\, S + \mu\, T)$.

2. Let $N = \max\{\, n : \mathbb{N} \mid a_n \neq b_n \,\} + 1$. $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet T_n - T_N = S_n - S_n$. Hence, $T_n = S_n + (T_N - S_N)$ for all but finitely many n. By Proposition 1.2.4.2, S_n converges iff T_n converges.

□

Definition 1.6.6. *A real sequence $(a_n)_{n:\mathbb{N}}$ is* ***non-negative*** *if it is bounded-below by 0, i.e. $\forall\, n : \mathbb{N} \bullet a_n \geqslant 0$.*

Proposition 1.6.7. *For a non-negative sequence $(a_n)_{n:\mathbb{N}}$,*

1. $(S_n)_{n:\mathbb{N}}$ is non-decreasing.

2. $\sum_{n=0}^{\infty} a_n$ converges iff $(S_n)_{n:\mathbb{N}}$ is bounded-above, otherwise it diverges to ∞.

Proof.

1. $\forall\, n : \mathbb{N} \bullet S_{n+1} - S_n = a_{n+1} \geqslant 0$.

2. If $\lim_{n\to\infty} S_n$ converges, then by Lemma 1.3.1, $(S_n)_{n:\mathbb{N}}$ is bounded-above. Conversely, if $(S_n)_{n:\mathbb{N}}$ is bounded-above, then by Theorem 1.4.2 and (1), it converges.

   Otherwise, by Theorem 1.4.2.2 and (1), $(S_n)_{n:\mathbb{N}}$ diverges to ∞.

□

Similarly, if $\forall\, n : \mathbb{N} \bullet a_n \leqslant 0$, then $(S_n)_{n:\mathbb{N}}$ is non-increasing, and $\lim\limits_{n\to\infty} S_n$ either converges, or diverges to $-\infty$.

1.7 Convergence Tests for Real Series

Theorem 1.7.1 (Comparison Test)**.** *For non-negative sequences $(a_n)_{n:\mathbb{N}}$, $(b_n)_{n:\mathbb{N}}$, and $\lambda : \mathbb{R}$, if $\forall\, n : \mathbb{N} \bullet a_n \leqslant \lambda\, b_n$, and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. Let $(S_n)_{n:\mathbb{N}}$ be the partial sums for $(a_n)_{n:\mathbb{N}}$, and $(T_n)_{n:\mathbb{N}}$ be the partial sums for $(b_n)_{n:\mathbb{N}}$. If $T_n \longrightarrow T$, then $\forall\, n : \mathbb{N} \bullet S_n \leqslant \lambda\, T_n \leqslant \lambda\, T$, so by Proposition 1.6.7.2, $\sum_{n=0}^{\infty} a_n$ converges. □

Theorem 1.7.2 (Alternating Series Test)**.** *If* $(a_n)_{n:\mathbb{N}}$ *is non-negative and non-increasing, and* $a_n \longrightarrow 0$*, then* $\displaystyle\sum_{k=0}^{\infty}(-1)^n a_n$ *converges.*

Proof. Let $S_n = \sum_{k=0}^{n}(-1)^k a_k$. $(a_{2n} - a_{2n+1})_{n:\mathbb{N}}$ and $(a_{2n+1} - a_{2n+2})_{n:\mathbb{N}}$ are non-negative, so by Proposition 1.6.7.1, their respective partial sums $(S_{2n+1})_{n:\mathbb{N}}$ and $(a_0 - S_{2n+2})_{n:\mathbb{N}}$ are non-negative and non-decreasing.

$\forall\, n : \mathbb{N}$, $S_{2n+1} = a_0 - (a_0 - S_{2(n-1)+2}) - a_{2n+1} \leqslant a_0$, so $(S_{2n+1})_{n:\mathbb{N}}$ is bounded-above. By Theorem 1.4.2, $S_{2n+1} \longrightarrow S$ for some $S : \mathbb{R}$. Then, $S_{2n} = S_{2n+1} - a_{2n+1} \longrightarrow S - 0 = S$, and hence $S_n \longrightarrow S$.[37] □

Theorem 1.7.3 (Root Test)**.** *For a non-negative sequence* $(a_n)_{n:\mathbb{N}}$*, if* $l = \lim_{n\to\infty} \sqrt[n]{a_n}$ *exists, then*

1. *If* $l < 1$*, then* $\sum_{n=0}^{\infty} a_n$ *converges.*
2. *If* $l > 1$*, then* $\sum_{n=0}^{\infty} a_n$ *diverges to* ∞*.*

Proof. Let $x = \frac{l+1}{2}$ and $\epsilon = |x - l| > 0$ for $l \neq 1$.

1. Suppose $l < 1$. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet \sqrt[n]{a_n} < l + \epsilon = x$. Hence, $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n < x^n$, with $x < 1$.
2. Suppose $l > 1$. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet \sqrt[n]{a_n} > l - \epsilon = x$. Hence, $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n > x^n$, with $x > 1$.

The result follows by Proposition 1.6.3, Proposition 1.6.5.2, and Theorem 1.7.1. □

Note that $\forall\, n : \mathbb{N} \bullet \sqrt[n]{a_n} \leqslant \sup\{\, k : \mathbb{N} \mid k \geqslant n \bullet \sqrt[n]{a_n}\,\}$, so if $\limsup \sqrt[n]{a_n} < 1$ then by Theorem 1.7.1, $\sum_{n=0}^{\infty} a_n$ converges.

Also, by Lemma 1.5.3.4, if $\limsup \sqrt[n]{a_n} > 1$ then $(a_n)_{n:\mathbb{N}}$ is unbounded-above, so $\sum_{n=0}^{\infty} a_n$ diverges to ∞.

Therefore, the same theorem holds for $l = \limsup \sqrt[n]{a_n}$.

Theorem 1.7.4 (D'Alembert's Ratio Test)**.** *For a non-negative sequence* $(a_n)_{n:\mathbb{N}}$*, if* $l = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ *exists, then*

1. *If* $l < 1$*, then* $\sum_{n=0}^{\infty} a_n$ *converges.*
2. *If* $l > 1$*, then* $\sum_{n=0}^{\infty} a_n$ *diverges to* ∞*.*

[37] I.e. given $\epsilon > 0$, take N, N' for $(S_{2n})_{n:\mathbb{N}}, (S_{2n+1})_{n:\mathbb{N}}$, then $\forall\, n : \mathbb{N} \mid n \geqslant \max\{N, N'\}$, $n = 2m$ or $n = 2m + 1$ for some $m : \mathbb{N}$, so either way $|S_n - S| < \epsilon$.

Proof. Let $x = \frac{l+1}{2}$ and $\epsilon = |x - l| > 0$ for $l \neq 1$.

1. Suppose $l < 1$. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet \dfrac{a_{n+1}}{a_n} < l + \epsilon = x$. Hence, $\forall\, n : \mathbb{N} \mid n \geqslant N,\ a_{n+1} < x a_n$, so $a_n = a_N \displaystyle\prod_{k=N}^{n-1} \frac{a_{k+1}}{a_k} < (x^{-N} a_N) x^n$.

2. Suppose $l > 1$. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet \dfrac{a_{n+1}}{a_n} > l - \epsilon = x$. Hence, $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_{n+1} > x a_n$, so $a_n = a_N \displaystyle\prod_{k=N}^{n-1} \frac{a_{k+1}}{a_k} > (x^{-N} a_N) x^n$.

The result follows by Proposition 1.6.3, Proposition 1.6.5.2, and Theorem 1.7.1. □

Theorem 1.7.5 (Cauchy's Condensation Test)**.** *For a non-negative, non-increasing sequence* $(a_n)_{n:\mathbb{N}}$, $\displaystyle\sum_{n=0}^{\infty} a_n$ *converges iff* $\displaystyle\sum_{n=0}^{\infty} 2^n a_{2^n}$ *converges.*

Proof. Let S_n and T_n be the partial sums.
$\forall\, n : \mathbb{N}$, let $m_n = \min\{\, k : \mathbb{N} \mid n < 2^k \,\} - 1$, then if $\sum_{n=0}^{\infty} 2^n a_{2^n} = T$,

$$S_n - a_0 = \sum_{k=1}^{n} a_k \leqslant \sum_{k=1}^{2^{m_n+1}-1} a_k = \sum_{k=0}^{m_n} \sum_{j=0}^{2^k-1} a_{2^k+j} \leqslant \sum_{k=0}^{m_n} 2^k a_{2^k} = T_{m_n} \leqslant T$$

so $(S_n)_{n:\mathbb{N}}$ is bounded-above. By Proposition 1.6.7.2, $\sum_{n=0}^{\infty} a_n$ converges.

Conversely, $\forall\, n : \mathbb{N}$, let $M_n = 2^{n+2} - 1$, then if $\sum_{n=0}^{\infty} a_n = S$,

$$\frac{T_n}{2} = \sum_{k=0}^{n} 2^{k-1} a_{2^k} = \sum_{k=1}^{n+1} 2^k a_{2^{k+1}} \leqslant \sum_{k=1}^{n+1} \sum_{j=0}^{2^k-1} a_{2^k+j} = \sum_{k=1}^{M_n} a_n \leqslant S_{M_n} \leqslant S$$

so $(T_n)_{n:\mathbb{N}}$ is bounded-above. By Proposition 1.6.7.2, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. □

1.8 Absolute and Conditional Convergence

Definition 1.8.1.

1. *A series* $\displaystyle\sum_{n=0}^{\infty} a_n$ *is* ***absolutely convergent*** *if* $\displaystyle\sum_{n=0}^{\infty} |a_n|$ *is convergent.*

2. *A series is* ***conditionally convergent*** *if it is convergent but not absolutely convergent.*

We also say a series "converges absolutely" or "converges conditionally". Every convergent non-negative series is absolutely convergent.

Proposition 1.8.2. *Every absolutely convergent series is convergent.*[38]

Proof. If $\sum_{n=0}^{\infty} |a_n|$ converges, then by Theorem 1.6.4, $\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, N : \mathbb{N} \bullet$ $\forall\, m, n : \mathbb{N} \mid n \geqslant m \geqslant N$,

$$\left|\sum_{k=m}^{n} a_n\right| \leqslant \sum_{k=m}^{n} |a_n| = \left|\sum_{k=m}^{n} |a_n|\right| < \epsilon$$

and so $\sum_{n=0}^{\infty} a_n$ converges. □

Definition 1.8.3. $\sum_{n=0}^{\infty} b_n$ *is a* ***rearrangement*** *of* $\sum_{n=0}^{\infty} a_n$ *if* $\exists\, f : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N} \bullet \forall\, n : \mathbb{N} \bullet b_n = a_{f(n)}$.

Note that "is a rearrangement of" is an equivalence relation.

Theorem 1.8.4.

1. *If* $\sum_{n=0}^{\infty} a_n$ *is absolutely convergent, and* $\sum_{n=0}^{\infty} b_n$ *is a rearrangement, then* $\sum_{n=0}^{\infty} b_n$ *is absolutely convergent, and* $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$.

2. *If a real series* $\sum_{n=0}^{\infty} a_n$ *is conditionally convergent, then* $\forall\, r : \mathbb{R}$*, there is a rearrangement which converges to* r*.*

Proof.

1. Let $S_n = \sum_{k=0}^{n} |a_k|$, $T_n = \sum_{k=0}^{n} |b_k|$, and $\forall\, n : \mathbb{N} \bullet b_n = a_{f(n)}$.
   Given $n : \mathbb{N}$, let $M = \max\{\, k : \mathbb{N} \mid k \leqslant n \bullet f^{-1}(k) \,\}$, then $T_n \leqslant S_M$. If $S_n \longrightarrow S$, then $(T_n)_{n:\mathbb{N}}$ is bounded-above by S, so by Proposition 1.6.7.2, $T_n \longrightarrow T \leqslant S$ for some $T : \mathbb{R}$. Hence, $\sum_{n=0}^{\infty} b_n$ is absolutely convergent.

[38] It would be a bit silly to call it "absolute convergence" otherwise, but we do have to check this.

Conversely, $\sum_{n=0}^{\infty} a_n$ is a rearrangement of $\sum_{n=0}^{\infty} b_n$, so $S \leqslant T$. Therefore, $S = T$, so the result holds for non-negative sequences.[39]

Suppose $(a_n)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}$ are real sequences.

Let $\alpha_n = |a_n| + a_n$, $\beta_n = |a_n| - a_n$, $\gamma_n = |b_n| + b_n$, and $\delta_n = |b_n| - b_n$. $(\alpha_n)_{n:\mathbb{N}}, (\beta_n)_{n:\mathbb{N}}, (\gamma_n)_{n:\mathbb{N}}, (\delta_n)_{n:\mathbb{N}}$ are non-negative, and each converges absolutely to some $\alpha, \beta, \gamma, \delta : \mathbb{R}$ respectively.[40] $(\gamma_n)_{n:\mathbb{N}}, (\delta_n)_{n:\mathbb{N}}$ are rearrangements of $(\alpha_n)_{n:\mathbb{N}}, (\beta_n)_{n:\mathbb{N}}$ respectively, so by the result for non-negative sequences, $\alpha = \gamma$ and $\beta = \delta$. Therefore,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{\alpha_n - \beta_n}{2} = \frac{\alpha - \beta}{2} = \frac{\gamma - \delta}{2} = \sum_{n=0}^{\infty} \frac{\gamma_n - \delta_n}{2} = \sum_{n=0}^{\infty} b_n$$

so the result holds for real sequences.

For complex sequences $(a_n)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}$, as before, if $(a_n)_{n:\mathbb{N}}$ converges absolutely then $(b_n)_{n:\mathbb{N}}$ converges absolutely, and so $(\mathrm{Re}(a_n))_{n:\mathbb{N}}$, $(\mathrm{Im}(a_n))_{n:\mathbb{N}}$, $(\mathrm{Re}(b_n))_{n:\mathbb{N}}$, and $(\mathrm{Im}(b_n))_{n:\mathbb{N}}$ converge absolutely. Proposition 1.6.5.1 and the result for real sequences,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n &= \left[\sum_{n=0}^{\infty} \mathrm{Re}(a_n)\right] + \mathrm{i}\left[\sum_{n=0}^{\infty} \mathrm{Im}(a_n)\right] \\ &= \left[\sum_{n=0}^{\infty} \mathrm{Re}(b_n)\right] + \mathrm{i}\left[\sum_{n=0}^{\infty} \mathrm{Im}(b_n)\right] = \sum_{n=0}^{\infty} b_n\end{aligned}$$

2. Let $(p_n)_{n:\mathbb{N}}, (q_n)_{n:\mathbb{N}}$ be the subsequences of non-negative and negative terms of $(a_n)_{n:\mathbb{N}}$ respectively,[41] and $(P_n)_{n:\mathbb{N}}, (Q_n)_{n:\mathbb{N}}$ be their partial sums. Suppose either $P_n \longrightarrow P$ or $Q_n \longrightarrow Q$ converges.

   If both converge, then the partial sums are bounded-above by P and bounded-below by Q respectively, and so the partial sums of $(|a_n|)_{n:\mathbb{N}}$ are bounded-above by $|P|+|Q|$. Hence by Proposition 1.6.7, $\sum_{n=0}^{\infty} |a_n|$ converges, a contradiction.

   If e.g. $P_n \longrightarrow P$ converges but $(Q_n)_{n:\mathbb{N}}$ diverges to $-\infty$, so that $\forall\, x : \mathbb{R} \bullet \exists\, n : \mathbb{N} \bullet Q_n < x$, then if $(S_n)_{n:\mathbb{N}}$ are the partial sums of $(a_n)_{n:\mathbb{N}}$,

[39] As $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |b_n| = \sum_{n=0}^{\infty} b_n$.

[40] By Theorem 1.7.1, and the fact that $|\alpha_n|, |\beta_n| \leqslant 2|a_n|$, and $|\gamma_n|, |\delta_n| \leqslant 2|b_n|$.

[41] $(a_n)_{n:\mathbb{N}}$ has infinitely many positive terms and infinitely many negative terms, otherwise by Proposition 1.6.5.2, $\sum_{n=0}^{\infty} |a_n|$ converges.

$\exists\, m : \mathbb{N} \bullet S_m = P_{m-n} + Q_n < P + x$, and so $(S_n)_{n:\mathbb{N}}$ is unbounded-below, a contradiction.

Therefore, both $(P_n)_{n:\mathbb{N}}, (Q_n)_{n:\mathbb{N}}$ diverge to ∞.

Define $(b_n)_{n:\mathbb{N}}, (i_n)_{n:\mathbb{N}}, (j_n)_{n:\mathbb{N}}$ inductively. Let $i_0 = j_0 = 0$, and for $n : \mathbb{N}$, if $\forall\, k : \mathbb{N} \mid k < n,\ b_k$ is defined, then

- If $\sum_{k=0}^{n-1} b_k < r$, let $b_n = p_{i_n}$, $i_{n+1} = i_n + 1$, and $j_{n+1} = j_n$.
- If $\sum_{k=0}^{n-1} b_k \geqslant r$, let $b_n = q_{j_n}$, $i_{n+1} = i_n$, and $j_{n+1} = j_n + 1$.

and let $(T_n)_{n:\mathbb{N}}$ be the partial sums of $(b_n)_{n:\mathbb{N}}$.

$\forall\, n : \mathbb{N} \bullet i_n + j_n = n$, so $(i_n + j_n) \longrightarrow \infty$. Suppose e.g. $(j_n)_{n:\mathbb{N}}$ converges to some j,[42] i.e. $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet j_n = j$.[43] Then $\forall\, n : \mathbb{N} \mid n \geqslant N$, $j_{n+1} = j_n$ so $T_{n-1} < r$, but also $T_n = P_{n-j} + Q_j$ is not bounded-above, a contradiction.

Therefore $i_n, j_n \longrightarrow \infty$, so

- $(b_n)_{n:\mathbb{N}}$ is a rearrangement of $(a_n)_{n:\mathbb{N}}$.[44]
- $\forall\, N : \mathbb{N} \bullet \exists\, m, n : \mathbb{N} \mid m, n > N \bullet (T_m < r) \wedge (T_n \geqslant r)$.[45]

By Lemma 1.6.2, $a_n \longrightarrow 0$, so by Proposition 1.2.4.4, $p_n, q_n \longrightarrow 0$. Hence, given $\epsilon > 0$, $\exists\, M : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant M \bullet |p_n|, |q_n| < \epsilon$.[46] Then e.g. if $T_M < r$,[47] take $N : \mathbb{N} \mid N > M$ so that $T_{N-1} < r$ and $T_N \geqslant r$.[48] $|T_N - r| \leqslant |T_N - T_{N-1}| = |b_N| < \epsilon$. Also, suppose $\exists\, n : \mathbb{N} \mid n \geqslant N$ so that, e.g. $T_n - r \geqslant \epsilon$,[49] with n minimal, so $T_{n-1} < T_n$. Then $0 < T_n - T_{n-1} = b_n < \epsilon$ and so $T_{n-1} > r$, contradicting $b_n > 0$.[50]

Therefore, $\forall\, n : \mathbb{N} \mid n \geqslant N \bullet |T_n - r| < \epsilon$, and so $T_n \longrightarrow r$.

$\square$

[42] $i_n \longrightarrow i$ is similar.

[43] This follows e.g. by taking $\epsilon = \frac{1}{2}$, as $(j_n)_{n:\mathbb{N}}$ only takes values from $\mathbb{N}$.

[44] Surjectivity: given $m : \mathbb{N}$, let $n = \min\{\, k : \mathbb{N} \mid i_k = n \,\}$, then $b_n = p_m$. Injectivity: the p_m term appears at position b_n when $m = i_n = i_{n+1} - 1$. $(i_n)_{n:\mathbb{N}}$ is non-decreasing, so $\forall\, n' : \mathbb{N} \mid n' > n \bullet i_{n'} \geqslant i_{n+1} > i_n$. Similarly for q_j.

[45] I.e. the partial sums continue alternating either side of r. This follows because for some $n : \mathbb{N}$, $b_n = q_{j_N+1}$, with $n > N$, $b_n < 0$ and hence $T_n \geqslant r$. Similarly for m.

[46] Take N', N'' for $(p_n)_{n:\mathbb{N}}, (q_n)_{n:\mathbb{N}}$, and $M = \max\{N', N''\}$.

[47] $T_M \geqslant r$ is similar.

[48] E.g. $N = \min\{\, k : \mathbb{N} \mid (k > M) \wedge (T_k \geqslant r) \,\}$.

[49] $r - T_n \geqslant \epsilon$ is similar.

[50] I.e. if $T_{n-1} > r$ then we would have chosen $b_n < 0$.

Proposition 1.8.5. *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are absolutely convergent, then* $\sum_{m,n=0}^{\infty} a_m b_n$ *converges absolutely, and*

$$\sum_{m,n=0}^{\infty} a_m b_n = \left[\sum_{n=0}^{\infty} a_n\right] \cdot \left[\sum_{n=0}^{\infty} b_n\right]$$

Proof. Let $(c_1, c_2) : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N}^2$ be an enumeration, and for $n : \mathbb{N}$, let

- $S_n = \sum_{k=0}^{n} a_n$, and $S'_n = \sum_{k=0}^{n} |a_n|$,
- $T_n = \sum_{k=0}^{n} b_n$, and $T'_n = \sum_{k=0}^{n} |b_n|$,
- $P_n = \sum_{k=0}^{n} a_{c_1(n)} b_{c_2(n)}$, and $P'_n = \sum_{k=0}^{n} \left|a_{c_1(n)} b_{c_2(n)}\right|$,

and let $S_n \longrightarrow S$, $S'_n \longrightarrow S'$, $T_n \longrightarrow T$, and $T'_n \longrightarrow T'$.

$\forall\, n : \mathbb{N} \bullet \exists\, M, N : \mathbb{N} \bullet P'_n < S'_M T'_N \leqslant S' T'$,[51] so by Proposition 1.6.7.2, $\sum_{m,n=0}^{\infty} a_m b_n$ converges absolutely.

Therefore by Theorem 1.8.4.1, the series can be evaluated in any ordering. Let (c_1, c_2) be an ordering for which $\forall\, k, n : \mathbb{N} \mid k < n^2 \bullet c_1(k), c_2(k) < n$.[52] Then $P_{n^2-1} = S_{n-1} T_{n-1} \longrightarrow ST$, so by Proposition 1.2.4.4, $P_n \longrightarrow ST$. □

1.9 Convergence Tests for Complex Series

Theorem 1.9.1 (Dirichlet's Test). *If* $(a_n)_{n:\mathbb{N}}$ *is a non-negative, non-increasing real sequence with* $a_n \longrightarrow 0$*, and* $(b_n)_{n:\mathbb{N}}$ *is a complex sequence with bounded partial sums, then* $\sum_{n=0}^{\infty} a_n b_n$ *converges.*[53]

Proof. Let $(T_n)_{n:\mathbb{N}}$ be the partial sums for $(b_n)_{n:\mathbb{N}}$. By assumption, $\exists\, B : \mathbb{R} \bullet \forall\, n : \mathbb{N} \bullet |T_n| \leqslant B$. Then $\forall\, n : \mathbb{N}$, the partial sums

$$\sum_{k=0}^{n} |T_k(a_k - a_{k+1})| \leqslant B \sum_{k=0}^{n} |a_k - a_{k+1}| \leqslant B a_0$$

are bounded-above,[54] and so $\sum_{n=0}^{\infty} T_n(a_n - a_{n+1})$ converges absolutely to some limit L.

[51] E.g. take $M = \max\{\, k : \mathbb{N} \mid k \leqslant n \bullet c_1(n)\,\}$ and $N = \max\{\, k : \mathbb{N} \mid k \leqslant n \bullet c_2(n)\,\}$.

[52] Such an enumeration can be contructed by induction on n, or explicitly, e.g. $(0,0), (0,1), (1,0), (1,1), (0,2), (1,2), (2,2), (2,1), (2,0), (0,3) \ldots$

[53] $b_n = (-1)^n$ gives Theorem 1.7.2.

[54] $0 \leqslant a_{k+1} \leqslant a_k \leqslant a_0$, so $a_k - a_{k+1} \leqslant a_0 - a_{k+1} \leqslant a_0$.

By Proposition 1.3.5.2, $T_n a_{n+1} \longrightarrow 0$, and so[55]

$$\sum_{k=0}^{n} a_k b_k = \left[\sum_{k=0}^{n} T_k(a_k - a_{k+1})\right] + T_n a_{n+1} \longrightarrow L + 0 = L$$

□

Corollary 1.9.2 (Abel's Test)**.** *If $(a_n)_{n:\mathbb{N}}$ is a non-negative, non-increasing real sequence, and $\sum_{n=0}^{\infty} b_n$ is a convergent complex series, then $\sum_{n=0}^{\infty} a_n b_n$ converges.*

Proof. By Theorem 1.4.2, $a_n \longrightarrow a$ for some $a : \mathbb{R}$. Hence, $(a_n - a)_{n:\mathbb{N}}, (b_n)_{n:\mathbb{N}}$ satisfy Theorem 1.9.1, so $\left[\sum_{n=0}^{\infty}(a_n - a)b_n\right] + a\left[\sum_{n=0}^{\infty} b_n\right] = \sum_{n=0}^{\infty} a_n b_n$. □

1.10 Decimal Expansions

Definition 1.10.1.

1. *A **decimal expansion** is a pair $(s, (a_n)_{n:\mathbb{N}})$, where $s : \{+, -\}$, and $(a_n)_{n:\mathbb{N}}$ is a sequence of $\mathbb{N}$ with $\forall\, n : \mathbb{N}_+ \bullet 0 \leqslant a_n \leqslant 9$.*
2. *A decimal expansion is **proper** if $\nexists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n = 9$, and $\neg((s = -) \wedge \forall\, n : \mathbb{N} \bullet a_n = 0)$. Otherwise it is **improper**.*
3. *The **value** of a decimal expansion is $s\, a_0.a_1 a_2 \ldots = s \sum_{n=0}^{\infty} 10^{-n} a_n$.*

By Theorem 1.7.1 and Proposition 1.6.3, this series always converges. If $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n = 0$,[56] we will write $s\, a_0.a_1 a_2 \ldots a_{N-1}$.

Theorem 1.10.2. *Every real number has a proper decimal expansion.*[57]

Proof. Wlog $x : \mathbb{R} \mid x \geqslant 0$. Define $a_0 = \lfloor x \rfloor$,[58] $r_0 = x - a_0$, and for $n : \mathbb{N}$, where a_n, r_n are defined, $0 \leqslant r_n < 1$, so let $a_{n+1} = \lfloor 10 r_n \rfloor$ and $r_{n+1} = 10 r_n - a_{n+1}$.

Inductively, $\forall\, n : \mathbb{N} \bullet a_n$ is defined, and $0 \leqslant x - \sum_{k=0}^{n} 10^{-k} a_k < 10^{-n}$. Therefore by Theorem 1.3.4, the partial sums converge to x. □

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[55] If T_n is the "integral" of b_n, and $a_{n+1} - a_n$ is the "derivative" of a_n, then this is "integration by parts".

[56] Or equivalently, if $a_n \longrightarrow 0$.

[57] I.e. $\forall\, x : \mathbb{R} \mid x \geqslant 0 \bullet x$ is the value of some decimal expansion.

[58] For $x : \mathbb{R}$, $\lfloor x \rfloor = \max\{\, k : \mathbb{Z} \bullet k \leqslant x \,\}$.

Theorem 1.10.3. *Decimal expansions* $(s, (a_n)_{n:\mathbb{N}}), (s, (b_n)_{n:\mathbb{N}})$ *have the same value iff either* $\forall\, n : \mathbb{N} \bullet a_n = b_n$, *or* $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n < N \bullet a_n = b_n$, *wlog* $a_N = b_N + 1$,[59] *and* $\forall\, n : \mathbb{N} \mid n > N \bullet a_n = 0$ *and* $b_n = 9$.

Proof. Let $N = \min\{\, n : \mathbb{N} \mid a_n \neq b_n \,\}$, if it exists.

$\sum_{n=N}^{\infty} 10^{-n} a_n = \sum_{n=N}^{\infty} 10^{-n} b_n$, so by Theorem 1.7.1 and Proposition 1.6.3,

$$-1 = \frac{-9 \cdot 10^{-1}}{1 - 10^{-1}} \leqslant a_N - b_N = 10^N \sum_{n=N+1}^{\infty} 10^{-n}(b_n - a_n) \leqslant \frac{9 \cdot 10^{-1}}{1 - 10^{-1}} = 1$$

$a_N, b_N \in \mathbb{Z}$, and $a_N \neq b_N$, so wlog $a_N = b_N + 1$. Hence, equality holds on the right, so $\forall\, n : \mathbb{N} \mid n > N \bullet b_n - a_n = 9$, and the result follows. □

Therefore, every real number has a unique proper decimal expansion.[60]

Definition 1.10.4. *A decimal expansion* $(s, (a_n)_{n:\mathbb{N}})$ *is* ***recurrent*** *if*

$$\exists\, k, N \bullet \mathbb{N} \mid k > 0 \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet a_n = a_{n+k}$$

Theorem 1.10.5. $x : \mathbb{R}$ *has a recurrent decimal expansion iff* $x \in \mathbb{Q}$.

Proof. If $(s, (a_n)_{n:\mathbb{N}})$ is recurrent, then by Proposition 1.6.3,

$$\sum_{n=N}^{\infty} 10^{-n} a_n = 10^{-N} \sum_{n=0}^{\infty} 10^{-kn} \underbrace{\sum_{i=0}^{k-1} 10^{-i} a_{N+i}}_{(*)} = \frac{10^{-N} \cdot (*)}{1 - 10^{-k}} \in \mathbb{Q}$$

Conversely, given $x : \mathbb{Q}$, $\exists\, p : \mathbb{Z},\ q, N : \mathbb{N} \bullet x = \dfrac{p}{10^N q}$ and $10, q$ coprime, so $\exists\, k : \mathbb{N} \bullet 10^k \equiv 1 \pmod{q'}$.

Hence, $\exists\, a, b : \mathbb{Z} \mid 0 \leqslant b < 10^k - 1 \bullet x = 10^{-N}\left(a + \dfrac{b}{10^k - 1}\right)$. Writing

$$a = \sum_{i=0}^{N} 10^{N-i} a_i,\ b = \sum_{i=0}^{k-1} 10^{k-i-1} b_i$$

with $a_i, b_i : \mathbb{Z}$, $0 \leqslant a_i \leqslant 9$ for $1 \leqslant i \leqslant N$, and $0 \leqslant b_i \leqslant 9$ for $0 \leqslant i < k$, it follows that x has a recurrent decimal expansion

$$x = a_0.a_1 a_2 \ldots a_{N-1} b_0 b_1 \ldots b_{k-1} b_0 b_1 \ldots b_{k-1} b_0 \ldots$$

□

[59] If $b_N = a_N + 1$, then relabel.

[60] The other special case is $+0.00\ldots = -0.00\ldots$

2 Real and Complex Functions

2.1 Limits of Functions

Definition 2.1.1. *For $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C} \to \mathbb{C}$, $a, l : \mathbb{R}$ or $\mathbb{C}$,*

1. *"$f(x) \longrightarrow l$ as $x \longrightarrow a$" if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \text{ or } \mathbb{C} \mid 0 < |x - a| < \delta \bullet |f(x) - l| < \epsilon$$

   I.e. for any distance ϵ, however small, there is a sufficiently small distance δ that, if x is within δ of a, then $f(x)$ is within ϵ of l.

For $f : \mathbb{R} \to \mathbb{R}$, $a, l : \mathbb{R}$,

2. *"$f(x) \longrightarrow l$ as $x \longrightarrow \infty$" if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, M : \mathbb{R} \bullet \forall\, x : \mathbb{R} \mid x \geqslant M \bullet |f(x) - l| < \epsilon$$

   I.e. for any distance ϵ, however small, f eventually gets, and stays, within ϵ of l for sufficiently large x.

3. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow a$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet f(x) > m$$

   I.e. for any potential upper bound m, however large, there is a sufficiently small distance δ that, if x is within δ of a, then f exceeds m.

4. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, M : \mathbb{R} \bullet \forall\, x : \mathbb{R} \mid x \geqslant M \bullet f(x) > m$$

   I.e. for any bound m, however large, f eventually gets, and stays, above m for sufficiently large x.

For $f : \mathbb{C} \to \mathbb{C}$, $a, l : \mathbb{C}$,

5. *"$f(x) \longrightarrow l$ as $x \longrightarrow \infty$" if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, M : \mathbb{R} \bullet \forall\, x : \mathbb{C} \mid |x| \geqslant M \bullet |f(x) - l| < \epsilon$$

6. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow a$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{C} \mid 0 < |x - a| < \delta \bullet |f(x)| > m$$

7. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, M : \mathbb{R} \bullet \forall\, x : \mathbb{C} \mid |x| \geqslant M \bullet |f(x)| > m$$

For real functions, we will write $x \longrightarrow -\infty$ to mean $-x \longrightarrow \infty$, and $f(x) \longrightarrow -\infty$ to mean $-f(x) \longrightarrow \infty$.

If f is only defined on a proper subset S of $\mathbb{R}$ or $\mathbb{C}$, then $x : S$.

Note that for these limits to be defined, f must be defined on some open interval containing a.

Theorem 2.1.2. *$f(x) \longrightarrow l$ as $x \longrightarrow a$ iff for every sequence $(x_n)_{n:\mathbb{N}}$, if $x_n \longrightarrow a$ and $\forall\, n : \mathbb{N} \bullet x_n \neq a$, then $f(x_n) \longrightarrow l$.*

Proof. Given $(x_n)_{n:\mathbb{N}}$ and $\epsilon > 0$, $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet |f(x) - l| < \epsilon$. Then $\exists\, N : \mathbb{N} \bullet \forall\, n : \mathbb{N} \mid n \geqslant N \bullet 0 < |x_n - a| < \delta$, so $|f(x_n) - l| < \epsilon$.

Otherwise, suppose $f(x) \not\longrightarrow l$ as $x \longrightarrow a$. Then $\exists\, \epsilon : \mathbb{R}_+ \bullet \forall\, \delta : \mathbb{R}_+ \bullet \exists\, x : \mathbb{R}$ or $\mathbb{C} \mid 0 < |x - a| < \delta \bullet |f(x) - l| \geqslant \epsilon$. Construct a sequence $(x_n)_{n:\mathbb{N}}$ by taking x_n to be a counterexample for $\delta = \frac{1}{n}$. $\forall\, n : \mathbb{N}$, $|x_n - a| < \frac{1}{n}$ so $x_n \longrightarrow a$, but $|f(x_n) - l| \geqslant \epsilon$ so $f(x_n) \not\longrightarrow l$. □

Similar results for $f(x) \longrightarrow \pm\infty$ and/or $x \longrightarrow \pm\infty$ can be shown. It follows that most results for limits of sequences have valid analogies for limits of functions.

Proposition 2.1.3. *For $f, g : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C} \to \mathbb{C}$, $a, l, m : \mathbb{R}$ or $\mathbb{C}$,*

1. *If $f(x) \longrightarrow l$ and $f(x) \longrightarrow m$ as $x \longrightarrow a$, then $l = m$.*
2. *If $\forall\, x : \mathbb{R}$ or $\mathbb{C} \bullet f(x) = l$, then $\forall\, b : \mathbb{R}$ or $\mathbb{C} \bullet f(x) \longrightarrow l$ as $x \longrightarrow b$.*

If $f(x) \longrightarrow l$ and $g(x) \longrightarrow m$ as $x \longrightarrow a$, then

3. *$\forall\, \lambda, \mu : \mathbb{R}$ or $\mathbb{C} \bullet (\lambda f(x) + \mu\, g(x)) \longrightarrow (\lambda\, l + \mu\, m)$ as $x \longrightarrow a$.*[61]
4. *$f(x) \cdot g(x) \longrightarrow lm$ as $x \longrightarrow a$.*
5. *For $l \neq 0$, $\dfrac{1}{f(x)} \longrightarrow \dfrac{1}{l}$ as $x \longrightarrow a$.*

Proof. By Proposition 1.2.4 and Theorem 2.1.2. □

Definition 2.1.4. *If $f(x) \longrightarrow l$ as $x \longrightarrow a$, then $\lim\limits_{x \to a} f(x) = l$.*

By Proposition 2.1.3.1, this is well-defined.

[61] I.e. $\lim_{x \to a}$ is a linear operator on functions.

Theorem 2.1.5 (Squeeze Theorem)**.** *For $f, g, h : \mathbb{R} \to \mathbb{R}$, $a, l : \mathbb{R}$, if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$, and $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet f(x) \leqslant g(x) \leqslant h(x)$, then $\lim_{x\to a} g(x) = l$.*

Proof. By Theorem 1.3.4 and Theorem 2.1.2. □

Lemma 2.1.6. *If $\lim\limits_{x\to a} f(x) = b$, $\lim\limits_{y\to b} g(y) = l$, and*

$$\exists\, d : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \textit{ or } \mathbb{C} \mid 0 < |x - a| < d \bullet f(x) \neq b$$

then $\lim\limits_{x\to a} gf(x) = l$.

Proof. Given $\epsilon > 0$, $\exists\, \delta' : \mathbb{R}_+ \bullet \forall\, y : \mathbb{R} \mid 0 < |y - b| < \delta' \bullet |g(y) - l| < \epsilon$. Hence, $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet |f(x) - b| < \delta'$.

Therefore, $\forall\, x : \mathbb{R} \mid 0 < |x - a| < \min\{\delta, d\}$, $0 < |f(x) - b| < \delta'$ and so $|gf(x) - l| < \epsilon$. □

Similar results e.g. for $x \longrightarrow \infty$, $y \longrightarrow \infty$, or $g(y) \longrightarrow \infty$, can be shown.

Definition 2.1.7. *For $f : \mathbb{R} \to \mathbb{R}$, $a, l : \mathbb{R}$*

1. *"$f(x) \longrightarrow l$ as $x \longrightarrow a^+$" if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a < x < a + \delta \bullet |f(x) - l| < \epsilon$$

2. *"$f(x) \longrightarrow l$ as $x \longrightarrow a^-$" if*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a - \delta < x < a \bullet |f(x) - l| < \epsilon$$

3. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow a^+$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a < x < a + \delta \bullet f(x) > m$$

4. *"$f(x) \longrightarrow \infty$ as $x \longrightarrow a^-$" if*

$$\forall\, m : \mathbb{R} \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a - \delta < x < a \bullet f(x) > m$$

"As $x \longrightarrow a$" is equivalent to "as $x \longrightarrow a^+$ and as $x \longrightarrow a^-$".

(See Appendix A.2 for a table of function limit definitions.)

Definition 2.1.8. *For $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C} \to \mathbb{C}$, $o(f)$ is the set of functions g for which* $\lim_{x \to 0} \frac{g(x)}{f(x)} = 0$.

We will write $G(x, o(f)) = 0$ to mean $\exists\, g : o(f) \bullet G(x, g(x)) = 0$.[62]

2.2 Continuity

Definition 2.2.1. *For $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C} \to \mathbb{C}$, $a : \mathbb{R}$ or $\mathbb{C}$,*

1. *f is "**continuous** at a" if $\lim_{x \to a} f(x) = f(a)$, i.e.*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \textit{ or } \mathbb{C} \mid |x - a| < \delta \bullet |f(x) - f(a)| < \epsilon$$

2. *f is **continuous** if $\forall\, x : \mathbb{R}$ or $\mathbb{C} \bullet f$ is continuous at x.*

Proposition 2.2.2. *If f and g are continuous at a,*

1. *$\forall\, \lambda, \mu : \mathbb{R}$ or $\mathbb{C} \bullet \lambda f + \mu g$ is continuous at a.*
2. *$f \cdot g$ is continuous at a.*
3. *For $f(a) \neq 0$, $\frac{1}{f}$ is continuous at a.*

Proof. The results follow immediately from Proposition 2.1.3. □

In particular, if f and g are continuous, then $\lambda f + \mu g$ and $f \cdot g$ are continuous.

It can be shown that the polynomial functions $c(X) = 1$, $f(X) = X$ are continuous. By Proposition 2.2.2, it follows that all polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$ are continuous, and all rational functions[63] are continuous everywhere they are defined.

Theorem 2.2.3. *If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Proof. Given $\epsilon > 0$,

$$\exists\, \delta' : \mathbb{R}_+ \bullet \forall\, y : \mathbb{R} \text{ or } \mathbb{C} \mid |y - f(a)| < \delta' \bullet |g(y) - gf(a)| < \epsilon$$

Then

$$\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \text{ or } \mathbb{C} \mid |x - a| < \delta \bullet |f(x) - f(a)| < \delta'$$

so $|gf(x) - gf(a)| < \epsilon$. □

[62] I.e. $G = 0$ is any equation in x and $o(f)$.

[63] A **rational function** is of the form $\frac{p}{q}$ for polynomials p, q with no common roots. Technically, in general they are not functions, but partial functions, as they are undefined at the roots of their denominators.

Definition 2.2.4. *For* $f : \mathbb{R} \to \mathbb{R}$,

1. f *is "**right-continuous** at* $a : \mathbb{R}$*" if* $\lim_{x \to a^+} f(x) = f(a)$, *i.e.*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a \leqslant x < a + \delta \bullet |f(x) - f(a)| < \epsilon$$

2. f *is "**left-continuous** at* $a : \mathbb{R}$*" if* $\lim_{x \to a^-} f(x) = f(a)$, *i.e.*

$$\forall\, \epsilon : \mathbb{R}_+ \bullet \exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a - \delta < x \leqslant a \bullet |f(x) - f(a)| < \epsilon$$

3. f *is **right-continuous** if* $\forall\, x : \mathbb{R} \bullet f$ *is right-continuous at* x.

4. f *is **left-continuous** if* $\forall\, x : \mathbb{R} \bullet f$ *is left-continuous at* x.

f is continuous at x iff it is right-continuous and left-continuous at x, and f is continuous iff it is right-continuous and left-continuous.

2.3 Real Functions on Closed Bounded Intervals

(See Appendix A.1 for a table of interval definitions.)

Definition 2.3.1. $f : [a, b] \to \mathbb{R}$ *is continuous if it is right-continuous at* a, *left-continuous at* b, *and* $\forall\, x : (a, b)$, *continuous at* x.

Theorem 2.3.2 (Intermediate Value Theorem)**.** *If* $f : [a, b] \to \mathbb{R}$ *is continuous, and* $f(a) < 0 < f(b)$, *then* $\exists\, c : (a, b) \bullet f(c) = 0$.

Proof. Let $S = \{\, x : [a, b] \mid f(x) < 0 \,\}$. $a \in S$ and $b \notin S$, so S is non-empty and bounded-above. Let $c = \sup S$. Suppose $f(c) \neq 0$. Take $\delta > 0$ so that $\forall\, x : [a, b] \mid |x - c| < 2\delta \bullet |f(x) - f(c)| < |f(c)|$.

If $f(c) < 0$, then $f(c + \delta) < f(c) + |f(c)| = 0$, so $S \ni c + \delta > c$, a contradiction.

If $f(c) > 0$, then $\forall\, x : [a, b]$, if $c - \delta \leqslant x \leqslant c$ then $f(x) > f(c) - |f(c)| = 0$ and if $f(x) \geqslant 0$ for $x \geqslant c$. Hence, $c - \delta < c$ is an upper bound of S, a contradiction.

Therefore, $f(c) = 0$. □

Corollary 2.3.3. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* $\forall\, c, d : [a, b], y : \mathbb{R} \mid f(c) \leqslant y \leqslant f(d) \bullet \exists\, t : [0, 1] \bullet f((1 - t)c + td) = y$.

Proof. Wlog $c \leqslant d$.[64] Define $g : [c, d] \to \mathbb{R}$ by $g(x) = f(x) - y$.

$g(c) = f(c) - y \leqslant 0$ and $g(d) = f(d) - y \geqslant 0$. If $g(c) = 0$ then $t = 0$, and if $g(d) = 0$ then $t = 1$. Otherwise, if $g(c) < 0 < g(d)$ then by Theorem 2.3.2 $\exists\, x : (c, d) \bullet g(x) = 0$ and $f(x) = y$, so $t = \frac{x - c}{d - c}$. □

[64] Otherwise consider $f(-x)$.

Corollary 2.3.4. *Every continuous* $f : [0,1] \to [0,1]$ *has a fixed point.*[65]

Proof. Let $g(x) = x - f(x)$. $g(0) \leqslant 0$ and $g(1) \geqslant 0$, so by Corollary 2.3.3, $\exists\, c : [0,1] \bullet g(c) = 0$ and $f(c) = c$. □

Theorem 2.3.5 (Nameless Theorem)**.** *If* $f : [a,b] \to \mathbb{R}$ *is continuous, then* f *is bounded*[66] *and attains its bounds.*[67]

Proof. Suppose f is unbounded-above. Then $\forall\, n : \mathbb{N} \bullet \exists\, x_n : [a,b] \bullet f(x_n) > n$, so that $f(x_n) \longrightarrow \infty$. By Theorem 1.5.7, $(x_n)_{n:\mathbb{N}}$ has a convergent subsequence $(y_n)_{n:\mathbb{N}}$. Let $y = \lim_{n\to\infty} y_n$. Then by Theorem 2.1.2, $f(y_n) \longrightarrow f(y)$, but by Proposition 1.2.4.4, $f(y_n) \longrightarrow \infty$, a contradiction. Therefore, f is bounded-above.

Let $s = \sup \operatorname{ran} f$. By Proposition 1.3.6, there is a sequence $(f(x'_n))_{n:\mathbb{N}}$ converging to s. By Theorem 1.5.7, $(x'_n)_{n:\mathbb{N}}$ has a convergent subsequence $(y'_n)_{n:\mathbb{N}}$. Let $y' = \lim_{n\to\infty} y'_n$. By Theorem 2.1.2, $f(y'_n) \longrightarrow f(y')$, and by Proposition 1.2.4.4, $f(y'_n) \longrightarrow s$. Therefore, $f(y') = s$.

Similarly, f is bounded-below and attains its infimum.[68] □

Corollary 2.3.6. *If* $f : [a,b] \to \mathbb{R}$ *is continuous, then* $\operatorname{ran} f$ *is a closed bounded interval.*

Proof. By Theorem 2.3.5, $\exists\, c, d : [a,b] \bullet f(c) = \inf \operatorname{ran} f \wedge f(d) = \sup \operatorname{ran} f$, so $\operatorname{ran} f \subseteq [f(c), f(d)]$. Wlog $c \leqslant d$.[69]

By Corollary 2.3.3, $\forall\, y : [f(c), f(d)] \bullet \exists\, x : [c,d] \bullet f(x) = y$, so $[f(c), f(d)] \subseteq \operatorname{ran} f$.

Hence, $\operatorname{ran} f = [f(c), f(d)]$. □

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[65] I.e. $\exists\, c : [0,1] \bullet f(c) = c$.
[66] I.e. $\operatorname{ran} f$ is bounded.
[67] I.e. $\exists\, c, d : [a,b] \bullet f(c) = \inf \operatorname{ran} f \wedge f(d) = \sup \operatorname{ran} f$.
[68] Or, e.g. because $-f$ is bounded-above and attains its supremum.
[69] Otherwise, consider $f(-x)$.

2.4 Monotonic Functions

Definition 2.4.1. *For $S : \mathbb{P}\,\mathbb{R}$, let $f : S \to \mathbb{R}$.*[70]

1. *f is **non-decreasing** if $\forall\, x, y : S \mid x < y \bullet f(x) \leqslant f(y)$.*
2. *f is **non-increasing** if $\forall\, x, y : S \mid x < y \bullet f(x) \geqslant f(y)$.*
3. *f is **monotonic** if it is non-decreasing or non-increasing.*
4. *f is **strictly-increasing** if $\forall\, x, y : S \mid x < y \bullet f(x) < f(y)$.*
5. *f is **strictly-decreasing** if $\forall\, x, y : S \mid x < y \bullet f(x) > f(y)$.*
6. *f is **strictly-monotonic** if it is strictly-increasing or strictly-decreasing.*

Theorem 2.4.2. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is injective iff it is strictly-monotonic.*

Proof. Suppose f is not strictly-monotonic. Wlog $\exists\, c, d, e : [a, b] \mid c < d < e \bullet f(c) \leqslant f(d) \geqslant f(e)$.[71] If $f(c) = f(d)$ or $f(d) = f(e)$, then f is not injective. Otherwise, $f(c) < f(d) > f(e)$, so $\exists\, y : (f(c), f(d)) \cap (f(e), f(d))$. By Corollary 2.3.3, $\exists\, x : (c, d) \bullet f(x) = y$ and $\exists\, x' : (d, e) \bullet f(x') = y$, and so f is not injective.

The converse is trivial. □

Similarly, $f : \mathbb{R} \to \mathbb{R}$ is injective iff it is strictly-monotonic.

Corollary 2.4.3. *If $f : [a, b] \to \mathbb{R}$ is continuous and strictly-increasing, then f is a bijection $[a, b] \rightarrowtail\!\!\!\!\twoheadrightarrow [f(a), f(b)]$, and f^{-1} is continuous.*

Proof. f is strictly-increasing, so $\inf \operatorname{ran} f = f(a)$ and $\sup \operatorname{ran} f = f(b)$. By Corollary 2.3.3, f is surjective, and by Theorem 2.4.2, f is injective.

Note that f^{-1} is strictly-increasing.

$\forall\, f(c) : [f(a), f(b)],\ \epsilon : \mathbb{R}_+$, let $\delta = \min\{f(c) - f(c - \epsilon), f(c + \epsilon) - f(c)\}$. Then, $\forall\, f(x) : [f(a), f(b)] \mid |f(x) - f(c)| < \delta$,

- $f(x) - f(c) < f(c + \epsilon) - f(c)$, so $f(x) < f(c + \epsilon)$, and $x < c + \epsilon$.
- $f(c) - f(x) < f(c) - f(c - \epsilon)$, so $f(x) > f(c - \epsilon)$, and $x > c - \epsilon$.

Hence, $|x - c| < \epsilon$. Therefore, f^{-1} is continuous. □

Similarly, if f is continuous and strictly-decreasing then it is a bijection : $[a, b] \rightarrowtail\!\!\!\!\twoheadrightarrow [f(b), f(a)]$, and f^{-1} is continuous. It follows that for any continuous injection $f : \mathbb{R} \rightarrowtail \mathbb{R}$, f^{-1} is a continuous bijection : $\operatorname{ran} f \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{R}$.

[70] Note that, since a real sequence is a function : $\mathbb{N} \to \mathbb{R}$, and $\mathbb{N} \in \mathbb{P}\,\mathbb{R}$, this is a direct generalisation of Definition 1.4.1.

[71] For $f(c) \geqslant f(d) \leqslant f(e)$, consider $-f$.

3 Differentiation

3.1 Differentiability

Definition 3.1.1. *For* $f : \mathbb{R} \to \mathbb{R}$ *or* $\mathbb{C} \to \mathbb{C}$, $a : \mathbb{R}$ *or* $\mathbb{C}$, $S : \mathbb{P}\mathbb{R}$ *or* $\mathbb{P}\mathbb{C}$,

1. f *is "**differentiable at** a" if* $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ *exists.*[72]
2. f *is "**differentiable on** S" if* $\forall\, x : S \bullet f$ *is differentiable at* x.
3. f *is **differentiable** if* $\forall\, x : \mathbb{R}$ *or* $\mathbb{C} \bullet f$ *is differentiable at* x.
4. *If* f *is differentiable at* a, *then* $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.
5. *If* f *is differentiable, then the function* f' *is the **derivative** of* f.
6. *If* f *is differentiable, then the operator*[73] $\dfrac{\mathrm{d}}{\mathrm{d}x}$ *is given by* $\dfrac{\mathrm{d}}{\mathrm{d}x} f = f'$.

Note that, as before, for these limits to be defined, f must be defined on some open interval containing a.

We will write $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ to mean $\dfrac{\mathrm{d}}{\mathrm{d}x} f$, and $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}$ to mean $\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n$.

Proposition 3.1.2. *If* f *is differentiable at* a, *then* f *is continuous at* a.

Proof. Define $g(h) = \dfrac{f(x) - f(a)}{x - a}$, so that $\lim_{x \to a} g(x) = f'(a)$. Then

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \big[f(a) + (x - a) g(x)\big] \\ &= f(a) + \Big[\lim_{x \to a} (x - a)\Big] \cdot \Big[\lim_{x \to a} g(x)\Big] = f(a) + 0 \cdot f'(a) = f(a).\end{aligned}$$

□

In particular, every differentiable function is continuous.

Proposition 3.1.3. *If* f *and* g *are differentiable at* a,

1. $\forall\, \lambda, \mu : \mathbb{R}$ *or* $\mathbb{C} \bullet \lambda f + \mu g$ *is differentiable at* a, *and* $(\lambda f + \mu g)'(a) = \lambda f'(a) + \mu g'(a)$.[74]

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[72] Or equivalently, if $\lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.

[73] $\frac{\mathrm{d}}{\mathrm{d}x}$ is a partial function : $(\mathbb{R} \to \mathbb{R}) \nrightarrow (\mathbb{R} \to \mathbb{R})$ or $(\mathbb{C} \to \mathbb{C}) \nrightarrow (\mathbb{C} \to \mathbb{C})$.

[74] I.e. $\frac{\mathrm{d}}{\mathrm{d}x}$ is a linear operator on differentiable functions.

2. *$f \cdot g$ is differentiable at a, and $(f \cdot g)'(a) = f(a)'g(a) + f(a)g'(a)$.*

3. *For $f(a) \neq 0$, $\frac{1}{f}$ is differentiable at a, and $\left(\frac{1}{f}\right)'(a) = \frac{-f'(a)}{f(a)^2}$.*

Proof.

1. $$\lim_{x \to a} \frac{(\lambda f(x) + \mu\, g(x)) - (\lambda f(a) + \mu\, g(a))}{x - a}$$
$$= \lim_{x \to a} \left[\lambda \frac{f(x) - f(a)}{x - a} + \mu \frac{g(x) - g(a)}{x - a}\right]$$
$$= \lambda \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right] + \mu \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\right] = \lambda f'(a) + \mu\, g'(a).$$

2. $$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} g(x) + \frac{g(x) - g(a)}{x - a} f(a)\right]$$
$$= \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right] \cdot \left[\lim_{x \to a} g(x)\right] + \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\right] \cdot f(a)$$
$= f'(a)g(a) + f(a)g'(a)$, as by Proposition 3.1.2, g is continuous at a.

3. $$\lim_{x \to a} \frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x - a} = \lim_{x \to a} \left[\frac{f(a) - f(x)}{x - a} \cdot \frac{1}{f(a)f(x)}\right]$$
$$= \left[- \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right] \cdot \left[\lim_{x \to a} \frac{1}{f(a)f(x)}\right]$$
$= -f'(a) \cdot \frac{1}{f(a)^2}$, as by Proposition 3.1.2, f is continuous at a.

□

In particular, if f and g are differentiable, then $\lambda f + \mu\, g$ and $f \cdot g$ are differentiable.

It can be shown that the polynomial functions $c(X) = 1$, $f(X) = X$ are differentiable with derivatives 0 and 1 respectively. By Proposition 3.1.3, it follows that all polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$ are differentiable, and all rational functions are differentiable everywhere they are defined.

Theorem 3.1.4 (Chain Rule)**.** *If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*[75]

[75] This is more commonly stated as $\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x}$, but in general f and g need not be differentiable everywhere.

Proof. By Proposition 3.1.2, f is continuous at a, and g is continuous at $f(a)$. Let $(x_n)_{n:\mathbb{N}}$ converge to a, with $\forall\, n : \mathbb{N} \bullet x_n \neq a$, so that $f(x_n) \longrightarrow f(a)$.

If $f(x_n) \neq f(a)$ for all but finitely many $n : \mathbb{N}$, then by Proposition 1.2.4.2,

$$\begin{aligned}\lim_{n\to\infty} \frac{gf(x_n) - gf(a)}{x_n - a} &= \lim_{n\to\infty} \left[\frac{gf(x_n) - gf(a)}{f(x_n) - f(a)} \cdot \frac{f(x_n) - f(a)}{x - a}\right] \\ &= \left[\lim_{n\to\infty} \frac{gf(x_n) - gf(a)}{f(x_n) - f(a)}\right] \cdot \left[\lim_{n\to\infty} \frac{f(x_n) - f(a)}{x - a}\right] = g'(f(a)) \cdot f'(a)\end{aligned}$$

by Theorem 2.1.2, and so $(gf)'(a) = g'(f(a))f'(a)$.

If $f(x_n) = f(a)$ for all but finitely many $n : \mathbb{N}$, then by Proposition 1.2.4.2,

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n\to\infty} \frac{f(a) - f(a)}{x_n - a} = 0$$

so

$$\lim_{n\to\infty} \frac{gf(x_n) - gf(a)}{x_n - a} = \lim_{n\to\infty} \frac{g(a) - g(a)}{x - a} = 0 = g'(f(a)) \cdot f'(a)$$

hence by Theorem 2.1.2, the result holds.

Otherwise, there are infinitely many $x_n, x_{n'}$ with $f(x_n) \neq f(a) = f(x_{n'})$. Let $(p_n)_{n:\mathbb{N}}, (q_n)_{n:\mathbb{N}}$ be subsequences such that $\forall\, n : \mathbb{N} \bullet f(p_n) \neq f(a) = f(q_n)$. By Proposition 1.2.4.4,

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n\to\infty} \frac{f(q_n) - f(a)}{q_n - a} = \lim_{n\to\infty} \frac{f(a) - f(a)}{x_n - a} = 0$$

so as before,

$$\lim_{n\to\infty} \frac{gf(p_n) - gf(a)}{p_n - a} = g'(f(a))f'(a) = 0 = \lim_{n\to\infty} \frac{gf(q_n) - gf(a)}{q_n - a}$$

hence by Lemma 1.2.6, $\lim\limits_{n\to\infty} \dfrac{gf(x_n) - gf(a)}{x_n - a} = 0$, and by Theorem 2.1.2, the result holds. □

Proposition 3.1.5. *For $\phi : \mathbb{R}$ or $\mathbb{C}$, $f(a + h) = f(a) + \phi h + o(h)$ iff f is differentiable at a and $f'(a) = \phi$.*

Proof. $f(a + h) = f(a) + \phi h + o(h)$ iff $\lim\limits_{h\to 0} \left[\dfrac{f(a + h) - f(a)}{h} - \phi\right] = 0$, iff $\phi = \lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h}$, iff $\phi = f'(a)$. □

In particular, if f is differentiable, and $\phi(x)$ is a function, then $\phi = f'$ iff $\forall\, x : \mathbb{R}$ or $\mathbb{C} \bullet f(x + h) = f(x) + h\phi(x) + o(h)$.

3.2 Differentiable Real Functions

Definition 3.2.1. *For* $f : [a, b] \to \mathbb{R}$*,*

1. f *is differentiable at* a *if* $\lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a}$ *exists.*

2. f *is differentiable at* b *if* $\lim_{x \to b^-} \dfrac{f(x) - f(b)}{x - b}$ *exists.*

Where the limits exist, $f'(a)$ *and* $f'(b)$ *take those values.*[76]

Definition 3.2.2. *For* $f : \mathbb{R} \to \mathbb{R}$*,* $c : \mathbb{R}$*,*

1. c *is a* ***local minimum*** *of* f *if* $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid |x - c| < \delta \bullet f(x) \geqslant f(c)$.

2. c *is a* ***local maximum*** *of* f *if* $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid |x - c| < \delta \bullet f(x) \leqslant f(c)$.

3. c *is a* ***stationary point*** *of* f *if* f *is differentiable at* c*, and* $f'(c) = 0$.

Proposition 3.2.3. *If* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable at* $a : \mathbb{R}$*, and* $f'(a) > 0$*, then* $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a < x < a + \delta \bullet f(x) > f(a)$.

Proof. $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < f'(a)$, so $\frac{f(x)-f(a)}{x-a} > 0$ and if $x > a$, then $f(x) - f(a) > 0$. □

Note also that $\forall\, x : \mathbb{R} \mid a - \delta < x < a \bullet f(x) < f(a)$. Similarly, if $f'(a) < 0$ then $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid a < x < a + \delta \bullet f(x) < f(a)$, and $\forall\, x : \mathbb{R} \mid a - \delta < x < a \bullet f(x) > f(a)$.

It follows immediately that if a is a local minimum or maximum of f, and f is differentiable at a, then a is a stationary point of f.

Corollary 3.2.4 (Darboux's Theorem)**.** *If* $f : [a, b] \to \mathbb{R}$ *is differentiable, and* $f'(a) \leqslant f'(b)$*, then* $\forall\, m : [f'(a), f'(b)] \bullet \exists\, c : [a, b] \bullet f'(c) = m$.[77]

Proof. For $m = f'(a)$ or $m = f'(b)$, take $c = a$ or $c = b$.

Otherwise $f'(a) < y < f'(b)$. Define $g(x) = f(x) - mx$. By Proposition 3.1.2, g is continuous, so by Theorem 2.3.5, g is bounded and attains its bounds. Also, $\forall\, x : [a, b] \bullet g'(x) = f'(x) - m$, so $g'(a) < 0 < g'(b)$, so neither a nor b are stationary points of g. Hence by g has a maximum at some $c : (a, b)$. By Proposition 3.2.3, $g'(c) = 0$, and hence $f'(c) = m$. □

[76] By this definition, e.g. $|x|$ is differentiable on $[-1, 0]$ and $[0, 1]$, but not $[-1, 1]$.

[77] By Theorem 2.3.2, every continuous function has *Darboux's property*. However, f' is not necessarily continuous.

Theorem 3.2.5 (Rolle's Theorem)**.** *For $a < b$, if $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $f(a) = f(b)$, then $\exists\, c : (a, b) \bullet f'(c) = 0$.*

Proof. By Theorem 2.3.5, f is bounded, and $\exists\, c, c' : [a, b] \bullet f(c) = \inf f$ and $f(c') = \sup f$. $f(c') \leqslant f(a) = f(b) \leqslant f(c)$.

If $\forall\, x : (a, b) \bullet f(x) = f(a)$, then $f'(x) = 0$.

Otherwise, wlog $f(c) < f(a)$.[78] $c \in (a, b)$ is a local minimum of f, so by Proposition 3.2.3, c is a stationary point of f. □

Theorem 3.2.6 (Mean Value Theorem)**.** *For $a < b$, if $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b), then $\exists\, c : (a, b) \bullet f'(c) = \frac{f(b)-f(a)}{b-a}$.*[79]

Proof. Let $m = \frac{f(b)-f(a)}{b-a}$, and define $g(x) = f(x) - mx$. $g(a) = g(b)$, and $\forall\, x : (a, b) \bullet g'(x) = f'(x) - m$. By Theorem 3.2.5 $\exists\, c : (a, b) \bullet g'(c) = 0$, and so $f'(c) = m$. □

Corollary 3.2.7. *For $a < b$, if $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $\forall\, x : (a, b) \bullet f'(x) = 0$, then $\forall\, x : [a, b] \bullet f(x) = f(a)$.*[80]

Proof. Suppose $\exists\, x : [a, b] \bullet f(x) \neq f(a)$. By Theorem 3.2.6, $\exists\, c : (a, b) \bullet f'(c) = \frac{f(x)-f(a)}{x-a} \neq 0$, a contradiction. □

Corollary 3.2.8. *For $h : \mathbb{R}_+$, if $f : [a, a + h] \to \mathbb{R}$ is continuous, and differentiable on $(a, a + h)$, then $\exists\, t : (0, 1) \bullet f(a + h) = f(a) + hf'(a + th)$.*

Proof. By Theorem 3.2.6, $\exists\, c : (a, a + h) \bullet f'(c) = \frac{f(a+h)-f(a)}{(a+h)-a}$, so letting $t = \frac{c-a}{h}$, $hf'(a + th) = f(a + h) - f(a)$. □

Corollary 3.2.9 (Cauchy's Mean Value Theorem)**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous, and differentiable on (a, b), and $g(a) \neq g(b)$, then*

$$\exists\, c : (a, b) \bullet \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \text{ or } f'(c) = g'(c) = 0.^{81}$$

Proof. Define $\phi(x) = ((g(b) - g(a))f(x) - (f(b) - f(a))g(x)$. $\phi(a) = \phi(b)$, and $\forall\, x : (a, b) \bullet \phi'(x) = ((g(b) - g(a))f'(x) - (f(b) - f(a))g'(x)$. By Theorem 3.2.5, $\exists\, c : (a, b) \bullet \phi'(c) = 0$.

Hence, $((g(b)-g(a))f'(c) = (f(b)-f(a))g'(c)$, and the result follows. □

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[78] If $f(a) < f(c')$, then consider $-f$.

[79] I.e. every chord has a parallel tangent.

[80] A similar result applies to Complex functions.

[81] If $\forall\, x : [a, b] \bullet g(x) = x$, this yields Theorem 3.2.6, so it is a direct generalisation.

Corollary 3.2.10 (L'Hôpital's Rule)**.** *For* $f, g : \mathbb{R} \to \mathbb{R}$*, differentiable,* $a : \mathbb{R}$ *with* $f(a) = g(a) = 0$*, if* $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet g'(x) \neq 0$*, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \text{ if it exists.}$$

Proof. By Corollary 3.2.9,

$$\forall\, x : \mathbb{R} \mid a < x < a + \delta \bullet \exists\, c : (a, x) \bullet \frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}$$

Similarly for $a - \delta < x < a$. This defines a function $c(x)$, with $\forall\, x : \mathbb{R} \mid 0 < |x - a| < \delta \bullet |c(x) - a| < x - a$, so $\lim_{x \to a} c(x) = a$.[82] Hence, if the limit exists, then by Lemma 2.1.6,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f'(c(x))}{g'(c(x))} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

□

Similar results for $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to \infty}$ and $\lim\limits_{x \to -\infty}$ exist.

Corollary 3.2.11. *For* $a < b$*, if* $f : [a, b] \to \mathbb{R}$ *is continuous, and differentiable on* (a, b) *with* $\forall\, x : (a, b) \bullet f'(x) > 0$*, then* f *is strictly-increasing.*

Proof. Suppose $\exists\, p, q : [a, b] \mid p < q \bullet f(p) \geqslant f(q)$. By Theorem 3.2.6, $\exists\, c : (p, q) \bullet f'(c) = \frac{f(q) - f(p)}{q - p} \leqslant 0$, a contradiction. □

Similarly, if $\forall\, x : (a, b) \bullet f'(x) < 0$ then f is strictly-decreasing.

Theorem 3.2.12. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, and differentiable on* (a, b) *with* $\forall\, x : (a, b) \bullet f'(x) > 0$*, then* f *is a bijection* $: [a, b] \rightarrowtail\!\!\!\!\rightarrow [f(a), f(b)]$ *and* f^{-1} *is differentiable on* $(f(a), f(b))$ *with* $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$.[83]

Proof. By Corollary 3.2.11, f is strictly-increasing, and by Corollary 2.4.3, f is a bijection and f^{-1} is continuous on $[f(a), f(b)]$.

Given $t : (f(a), f(b))$, let $s = f^{-1}(t) \in (a, b)$. $f^{-1}(y) \longrightarrow s$ as $y \longrightarrow t$, and $f^{-1}(y) \neq s$ for $y \neq t$, so by Lemma 2.1.6,

$$\lim_{x \to s} \frac{x - s}{f(x) - f(s)} = \lim_{y \to t} \frac{f^{-1}(y) - s}{f(f^{-1}(y)) - t} = \lim_{y \to t} \frac{f^{-1}(y) - f^{-1}(t)}{y - t}$$

[82] Given $\epsilon > 0$, if $|x - a| < \epsilon$ then $|c(x) - a| < \epsilon$.

[83] This is more commonly written as $\dfrac{\mathrm{d}x}{\mathrm{d}f} = \dfrac{1}{\left(\dfrac{\mathrm{d}f}{\mathrm{d}x}\right)}$.

$$\text{and so } (f^{-1})'(t) = \lim_{y\to t} \frac{f^{-1}(y) - f^{-1}(t)}{y - t} = \lim_{x\to s} \frac{x - s}{f(x) - f(s)}$$
$$= \lim_{x\to s} \frac{1}{\left(\dfrac{f(x) - f(s)}{x - s}\right)} = \frac{1}{\lim\limits_{x\to s} \dfrac{f(x) - f(s)}{x - s}} = \frac{1}{f'(s)} = \frac{1}{f'(f^{-1}(t))}.$$

□

3.3 Convex Functions

Proposition 3.3.1. *For $f : \mathbb{R} \to \mathbb{R}$, $c : \mathbb{R}$, if $f'(c) = 0$ and $f''(c) > 0$ then $\exists\, \delta : \mathbb{R}_+ \bullet f$ is strictly-decreasing on $(c - \delta, c)$ and strictly-increasing on $(c, c + \delta)$.*

Proof. f' is differentiable at c, so f' is defined on $(c - d, c + d)$ for some $d > 0$. By Proposition 3.2.3, $\exists\, \delta : \mathbb{R}_+ \mid \delta \leqslant d \bullet \forall\, x : \mathbb{R} \mid c < x < c + \delta \bullet f'(x) > f'(c)$, so by Corollary 3.2.11, f is strictly-increasing on $(c, c + \delta)$. Similarly for $(c - \delta, c)$. □

In particular, since f is continuous at c, c is a local minimum of f. Similarly, if $f'(c) = 0$ and $f''(c) < 0$, f is strictly-increasing on $(c - \delta, c)$ and strictly-decreasing on $(c, c + \delta)$, and c is a local maximum of f.

Definition 3.3.2. *For $f : \mathbb{R} \to \mathbb{R}$,*

1. *f is **convex** if*

$$\forall\, a, b : \mathbb{R}, t : [0, 1] \mid a \leqslant b \bullet f((1 - t)a + tb) \leqslant (1 - t)f(a) + tf(b)$$

2. *f is **strictly-convex** if*

$$\forall\, a, b : \mathbb{R}, t : [0, 1] \mid a < b \bullet f((1 - t)a + tb) < (1 - t)f(a) + tf(b)$$

Proposition 3.3.3. *For $f : \mathbb{R} \to \mathbb{R}$, $\lambda, \mu : \mathbb{R}$, the function $f(x) + \lambda\, x + \mu$ is convex iff f is convex, and strictly-convex iff f is strictly-convex.*

Proof. Let $c = (1 - t)a + tb$, then

$$\begin{aligned} & f(c) + \lambda(c) + \mu \leqslant (1 - t)(f(a) + \lambda\, a + \mu) + t(f(b) + \lambda\, b + \mu) \\ \Leftrightarrow\ & f(c) + \lambda\, c + \mu \leqslant (1 - t)f(a) + tf(b) + \lambda\, c + \mu \\ \Leftrightarrow\ & f(c) \leqslant (1 - t)f(a) + tf(b). \end{aligned}$$

$<$ is similar. □

Theorem 3.3.4. *If $f : \mathbb{R} \to \mathbb{R}$ is twice-differentiable*[84]*, and $\forall\, x : \mathbb{R} \bullet f''(x) > 0$, then f is strictly-convex.*

Proof. Let $c = (1-t)a + tb$, define $g(x) = f(x) - f'(c)x$. $\forall\, x : (a,b) \bullet g''(c) = f''(c) > 0$, so by Corollary 3.2.11, g' is strictly-increasing. $g'(c) = 0$, so $\forall\, x : (a,c) \bullet g'(x) < 0$ and $\forall\, x : (c,b) \bullet g'(x) > 0$, hence g is strictly-decreasing on (a,c) and strictly-increasing on (c,b). By Proposition 3.1.2, f is continuous at a and b, so $g(a), g(b) > g(c)$.[85] Therefore,

$$\begin{aligned} g(c) &= (1-t)g(c) + tg(c) < (1-t)g(a) + tg(b) \\ \therefore f(c) - f'(c)c &< (1-t)(f(a) - f'(c)a) + t(f(b) - f'(c)b) \\ &= (1-t)f(a) + tf(b) - f'(c)((1-t)a + tb) \\ &= (1-t)f(a) + tf(b) - f'(c)c \\ \therefore f(c) &< (1-t)f(a) + tf(b) \end{aligned}$$

and so f is strictly-convex.[86] □

3.4 Higher Derivatives

For $n : \mathbb{N}$, where it exists, we will write $f^{(n)}$ to mean $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$.[87]

Proposition 3.4.1.

1. *Every monomial function aX^n is differentiable, with $\frac{\mathrm{d}}{\mathrm{d}X} aX^n = anX^{n-1}$.*
2. *The derivative of any polynomial function of degree $n > 0$ is a polynomial function of degree $n-1$.*[88]
3. *For a polynomial function f of degree n, $\forall\, k : \mathbb{N} \bullet f^{(k)}$ exists, and if $k \leqslant n$ then $f^{(k)}(0) = k!a_k$ where a_k is the coefficient of X^k in f, otherwise $f^{(k)}(0) = 0$.*

Proof.

[84] I.e. f is differentiable and f' is differentiable.

[85] This allows the slightly more general case where $\operatorname{dom} f = [\alpha, \beta]$ and f is not necessarily differentiable at α and β.

[86] While the reasoning is the same as in the proof of Proposition 3.3.3, we can't simply apply it to this particular g as we have only shown that it satisfies the equation for this particular t.

[87] In some texts, n is written using Roman numerals.

[88] The derivative of a polynomial of degree 0 is the constant polynomial 0, which has degree $-\infty$.

1. By induction on n. $n = 0, 1$ are trivial.

   For $n > 1$, by Proposition 3.1.3.2,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}X} aX^{n+1} = \frac{\mathrm{d}}{\mathrm{d}X}(aX^n \cdot X) &= \left[\frac{\mathrm{d}}{\mathrm{d}X} aX^n\right] \cdot X + aX^n \cdot \left[\frac{\mathrm{d}}{\mathrm{d}X} X\right] \\ &= anX^{n-1} \cdot X + aX^n \cdot 1 = a(n+1)X^n \end{aligned}$$

2. By Proposition 3.1.3.1 and (1), $\frac{\mathrm{d}}{\mathrm{d}X} \sum_{i=0}^{n} a_i X^i = \sum_{i=1}^{n} i a_i X^{i-1}$.

3. By (2), inductively for $0 \leqslant k \leqslant n$,

$$f^{(k)}(X) = \frac{\mathrm{d}^k}{\mathrm{d}X^k} \sum_{i=0}^{n} a_i X^i = \sum_{i=k}^{n} \frac{i!}{(i-k)!} a_i X^{i-k}$$

   Evaluating at $X = 0$, $f^{(k)}(0) = \frac{k!}{0!} a_k + \sum_{i=k+1}^{n} 0 = k! a_k$.

   By (2), for $k > n$, $f^{(k)}(X) = 0$ as a polynomial.

□

In particular, a polynomial can be uniquely determined by evaluating its derivatives at a single point.

Lemma 3.4.2. *If $f : [a, a+\delta) \to \mathbb{R}$ is $(n-1)$-times-differentiable,*[89] *then*

$$\forall\, h : (0, \delta) \bullet f(a+h) = \left[\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} h^k\right] + \frac{R_n(h)}{n!} h^n$$

for some function R_n, where

$$\forall\, h : (0, \delta) \bullet \exists\, c : (0, h) \bullet R_n(h) = \frac{f^{(n-1)}(a+c) - f^{(n-1)}(a)}{c}$$

Proof. For $u : [0, h]$, define

$$\phi(u) = f(a+u) - \left[\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} u^k\right] - \frac{R_n(h)}{n!} u^n$$

[89] I.e. $\forall\, k : \mathbb{N} \mid k \leqslant n-1 \bullet f^{(k)}$ exists.

- $\phi(0) = \phi(h) = 0$.
- By Proposition 3.4.1.3, ϕ is $(n-1)$-times-differentiable.
- $\forall\, k : \mathbb{N} \mid k < n-1 \bullet \phi^{(k)}(0) = 0$.
- $\phi^{(n-1)}(u) = f^{(n-1)}(a+u) - f^{(n-1)}(a) - R_n(h)u$.

Let $c_0 = h$, and inductively for $0 \leqslant k < n-1$, if c_k is defined, then $\phi^{(k)}(0) = \phi^{(k)}(c_k) = 0$ so by Theorem 3.2.5, $\exists\, c_{k+1} : (0, c_k) \bullet \phi^{(k+1)}(c_{k+1}) = 0$. Letting $c = c_{n-1}$, $\phi^{(n-1)}(c) = f^{(n-1)}(a+c) - f^{(n-1)}(a) - R_n(h)c$, and the result follows. □

R_n is the *remainder.*

Theorem 3.4.3. *If $f : [a, \delta) \to \mathbb{R}$ is n-times-differentiable at a, then*

$$f(a+h) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k\right] + o(h^n)$$

Proof. $f^{(n)}(a)$ exists, so wlog $f^{(n-1)}$ exists on $[a, a+\delta)$.[90] By Lemma 3.4.2,

$$\forall\, h : (0, \delta) \bullet \exists\, c : (0, h) \bullet R_n(h) = \frac{f^{(n-1)}(a+c) - f^{(n-1)}(a)}{c}$$

Considering $c(h)$ as a function, $\forall\, h : (0, \delta) \bullet 0 < c(h) < h$. By Lemma 2.1.6,

$$\begin{aligned} f^{(n)}(a) &= \lim_{h \to 0^+} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} \\ &= \lim_{h \to 0^+} \frac{f^{(n-1)}(a+c(h)) - f^{(n-1)}(a)}{c(h)} = \lim_{h \to 0^+} R_n(h) \end{aligned}$$

Hence $\dfrac{R_n(h)}{n!} h^n = \dfrac{f^{(n)}(a)}{n!} h^n + o(h^n)$. □

Theorem 3.4.4 (Taylor's Theorem)**.** *If $f : [a, a+\delta) \to \mathbb{R}$ is n-times-differentiable, then*

$$\forall\, h : (0, \delta) \bullet \exists\, t : (0, 1) \bullet f(a+h) = \left[\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} h^k\right] + \frac{f^{(n)}(a+th)}{n!} h^n$$

---

[90] $f^{(n-1)}$ exists on $[a, a+d)$ for some $d > 0$, so relabel $\delta = \min\{\delta, d\}$.

Proof. By Proposition 3.1.2, $f^{(n-1)}$ is continuous. By Lemma 3.4.2,

$$\forall\, h : (0, \delta) \bullet \exists\, c : (0, h) \bullet R_n(h) = \frac{f^{(n-1)}(a+c) - f^{(n-1)}(a)}{c}$$

Hence by Theorem 3.2.6, $\exists\, x : (0, c) \bullet f^{(n)}(x) = R_n(h)$. Letting $t = \dfrac{x-a}{h}$, the result follows. □

$a = 0$ gives *Maclaurin's Theorem.* $R_n(h) = f^{(n)}(a + th)$ is Lagrange's form for the remainder. Cauchy's form $R_n(h) = n(1-t)^{n-1} f^{(n)}(a+th)$ is also valid.[91]

Corollary 3.4.5. *If* $f : [a, a+\delta) \to \mathbb{R}$ *is infinitely differentiable,*[92] *then*

$$\forall\, h : (0, \delta) \bullet f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n \textit{ iff } \lim_{n\to\infty} \frac{R_n(h)}{n!} h^n = 0$$

Proof. Let $(S_n)_{n:\mathbb{N}}$ be the partial sums. By Lemma 3.4.2,

$$S_n = f(a+h) - \frac{R_{n+1}(h)}{(n+1)!} h^{n+1} \longrightarrow f(a+h) \text{ iff } \frac{R_{n+1}(h)}{(n+1)!} h^{n+1} \longrightarrow 0$$

□

In particular, the function f can be completely determined on some interval $(a-\delta, a+\delta)$ by evaluating its derivatives at a.[93]

$f(a+h) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n$ is the **Taylor series** of f at a.

3.5 Differentiable Complex Functions

Theorem 3.5.1. *For* $f : \mathbb{C} \to \mathbb{C}$, $a : \mathbb{C}$, *if* $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, z : \mathbb{C} \mid |z - a| < \delta \bullet$ f *is differentiable at* z, *then* f *is infinitely differentiable at* a, *and its Taylor series at* a *is valid* $\forall\, z : \mathbb{C} \mid |z-a| < \delta$.

Proof. Omitted.[94] □

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[91] See Corollary 5.5.3.2.

[92] I.e. $\forall\, n : \mathbb{N} \bullet f^{(n)}$ exists.

[93] Consider $f(-x)$ at $-a$.

[94] See *IB Complex Analysis.*

4 Power Series

4.1 Convergence of Power Series

Definition 4.1.1. *A* ***power series*** *is a function of X given by* $\displaystyle\sum_{n=0}^{\infty} a_n X^n$, *or* $\displaystyle\sum_{n=0}^{\infty} a_n (X - z_0)^n$, *where $(a_n)_{n:\mathbb{N}}$ is a complex sequence, and $z_0 : \mathbb{C}$.*

Definition 4.1.2. *The* ***radius of convergence*** *of a power series $\sum_{n=0}^{\infty} a_n X^n$ is $\sup\{\, z : \mathbb{C} \mid \sum_{n=0}^{\infty} a_n z^n$ converges $\bullet\ |z| \,\}$, or ∞ if the set is unbounded.*[95]

I.e. the radius of convergence is the radius of the smallest disc containing all the points at which the power series converges.

Theorem 4.1.3. *If $\sum_{n=0}^{\infty} a_n X^n$ has radius of convergence R, then for $z : \mathbb{C}$,*

1. *If $|z| < R$, the power series converges absolutely at z.*
2. *If $|z| > R$, the power series diverges.*

Proof.

1. $\forall\, z : \mathbb{C} \mid |z| < R \bullet \exists\, w : \mathbb{C} \mid |z| < |w| < R \bullet \sum_{n=0}^{\infty} a_n w^n$ converges. By Lemma 1.6.2, $a_n w^n \longrightarrow 0$, so by Lemma 1.3.1, $(|a_n w^n|)_{n:\mathbb{N}}$ has some upper bound B. Hence, by Theorem 1.7.1 and Proposition 1.6.3,
$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n w^n| \cdot \left|\frac{z}{w}\right|^n \leqslant B \sum_{n=0}^{\infty} \left|\frac{z}{w}\right|^n = \frac{B}{1 - \left|\frac{z}{w}\right|}$$
as $\left|\frac{z}{w}\right| < 1$. Hence, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.
2. Immediate from the definition of R.

□

Let $l = \limsup \sqrt[n]{|a_n|}$, if it exists. By Theorem 1.7.3, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $l\,|z| < 1$, and if $l\,|z| > 1$ then $a_n z^n \not\longrightarrow 0$ so by Lemma 1.6.2 the series diverges. It follows that $R = \frac{1}{l}$.

[95] The set is non-empty, as at $X = 0$, the power series converges to a_0. For $z : \mathbb{C}$, we will say $|z| < \infty$.

Proposition 4.1.4. *If* $\sum_{n=0}^{\infty} a_n X^n$*,* $\sum_{n=0}^{\infty} b_n X^n$ *have radii of convergence* R_1, R_2*, then the following power series have radii of convergence* $\geqslant \min\{R_1, R_2\}$*.*

1. $\forall\, \lambda, \mu : \mathbb{C} \bullet \lambda \left[\sum_{n=0}^{\infty} a_n X^n\right] + \mu \left[\sum_{n=0}^{\infty} b_n X^n\right] = \sum_{n=0}^{\infty} (\lambda\, a_n + \mu\, b_n) X^n.$

2. $\left[\sum_{n=0}^{\infty} a_n X^n\right] \cdot \left[\sum_{n=0}^{\infty} b_n X^n\right] = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_k + b_{n-k}\right] X^n.$

Proof. By Theorem 4.1.3, for $|z| < \min\{R_1, R_2\}$, $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converge absolutely.

The results follow by Proposition 1.6.5.1 and Proposition 1.8.5. □

4.2 Differentiation of Power Series

Lemma 4.2.1. $\sum_{n=0}^{\infty} a_n X^n$ *and* $\sum_{n=1}^{\infty} n a_n X^{n-1}$ *have the same radius of convergence.*

Proof. For $z : \mathbb{C}$, suppose $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely. Then for $n \geqslant 1$, $|a_n z^n| \leqslant |z| \cdot |n a_n z^{n-1}|$, so by Theorem 1.7.1, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Hence $R_1 \geqslant R_2$.

Conversely, if $|z| < R_1$ then $\exists\, w : \mathbb{C} \mid |z| < |w| < R_1$, so $\sum_{n=0}^{\infty} a_n w^n$ converges absolutely. $(n \left|\frac{z}{w}\right|^{n-1})_{n:\mathbb{N}}$ has some upper bound B,[96] hence by Theorem 1.7.1,

$$\sum_{n=1}^{\infty} \left|n a_n z^{n-1}\right| = \sum_{n=1}^{\infty} n \left|\frac{z}{w}\right|^{n-1} \left|a_n w^{n-1}\right| \leqslant \frac{B}{|w|} \sum_{n=0}^{\infty} |a_n w^n|$$

and so $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely. Therefore, $R_2 \geqslant R_1$. □

Theorem 4.2.2. $f(X) = \sum_{n=0}^{\infty} a_n X^n$ *with radius of convergence* R *is differentiable at* $X = z$ *if* $|z| < R$*, and* $f'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}$*.*[97]

[96] E.g. for $|x| < 1$, by Proposition 4.1.4.2 $\sum_{n=1}^{\infty} n x^{n-1} = \left(\sum_{n=0}^{\infty} x^n\right)^2$, so by Lemma 1.6.2, $n x^{n-1} \longrightarrow 0$, and by Lemma 1.3.1, $(n x^{n-1})_{n:\mathbb{N}}$ is bounded.

[97] I.e. $\frac{\mathrm{d}}{\mathrm{d}X} \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}X}$ on sequences of monomials $(a_n X^n)_{n:\mathbb{N}}$.

Proof. $\exists\, r : (|z|, R)$. By Lemma 4.2.1, $\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}$ converges, so given $\epsilon > 0$, $\exists\, N : \mathbb{N} \bullet \sum_{n=N}^{\infty} n\,|a_n|\,r^{n-1} < \frac{\epsilon}{3}$.[98]

$\forall\, h : \mathbb{C} \mid |z| + |h| < r$, by Theorem 3.2.6,

$$\exists\, t : (0,1) \bullet \left| \frac{(z+h)^n - z^n}{h} \right| = \left| n(z+th)^{n-1} \right| < nr^{n-1}$$

Therefore,

$$\overbrace{\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right|}^{(*)}$$

$$= \left| \frac{\left[\sum_{n=0}^{\infty} a_n (z+h)^n\right] - \left[\sum_{n=0}^{\infty} a_n z^n\right]}{h} - \left[\sum_{n=1}^{\infty} na_n z^{n-1}\right] \right|$$

$$= \left| \frac{\left[\sum_{n=0}^{N-1} a_n (z+h)^n\right] - \left[\sum_{n=0}^{N-1} a_n z^n\right]}{h} - \left[\sum_{n=1}^{N-1} na_n z^{n-1}\right] \right.$$
$$\left. + \left[\sum_{n=N}^{\infty} a_n \frac{(z+h)^n - z^h}{h}\right] - \left[\sum_{n=N}^{\infty} na_n z^{n-1}\right] \right|$$

$$\leqslant \underbrace{\left| \frac{\left[\sum_{n=0}^{N-1} a_n (z+h)^n\right] - \left[\sum_{n=0}^{N-1} a_n z^n\right]}{h} - \left[\sum_{n=1}^{N-1} na_n z^{n-1}\right] \right|}_{(\dagger)} + 2 \underbrace{\sum_{n=N}^{\infty} n\,|a_n|\,r^{n-1}}_{< \frac{2\epsilon}{3}}$$

By Proposition 3.4.1, $\frac{\mathrm{d}}{\mathrm{d}X} \sum_{n=0}^{N-1} a_n X^n = \sum_{n=1}^{N-1} na_n X^{n-1}$, so by Proposition 3.1.5, $(\dagger) \longrightarrow 0$ as $h \longrightarrow 0$, so $\exists\, h : \mathbb{C} \mid h \neq 0 \bullet (\dagger) < \frac{\epsilon}{3}$, so $(*) < \epsilon$, and hence $(*) \longrightarrow 0$ as $h \longrightarrow 0$. Therefore by Proposition 3.1.5, f is differentiable at z, and $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. □

It follows immediately that functions defined by power series are continuous and infinitely differentiable inside their radius of convergence, and equal their Taylor series.

[98] If the partial sums $S_n \longrightarrow S$, then $\exists\, N : \mathbb{N} \bullet S - S_{N-1} < \frac{\epsilon}{3}$.

4.3 The Exponential and Natural Logarithm Functions

Definition 4.3.1. $\exp : \mathbb{C} \to \mathbb{C}$ *is defined by* $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Proposition 4.3.2. *The radius of convergence of* exp *is* ∞.[99]

Proof. $\exp(0) = 1$, and

$$\forall\, z : \mathbb{C} \mid |z| > 0 \bullet \left| \frac{z^{n+1}}{(n+1)!} \right| \div \left| \frac{z^n}{n!} \right| = \frac{|z|}{n+1} \longrightarrow 0$$

so by Theorem 1.7.4, $\exp(z)$ converges absolutely. □

Lemma 4.3.3. $\forall\, z, w : \mathbb{C} \bullet \exp(z+w) = \exp(z) \cdot \exp(w)$.[100]

Proof. By Proposition 1.8.5 and Proposition 4.1.4.2,

$$\begin{aligned}
\exp(z) \cdot \exp(w) &= \left[\sum_{n=0}^{\infty} \frac{z^n}{n!}\right] \cdot \left[\sum_{n=0}^{\infty} \frac{w^n}{n!}\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n = exp(z+w)
\end{aligned}$$

□

Proposition 4.3.4.

1. $\exp' = \exp$.
2. $\forall\, z : \mathbb{C} \bullet \exp(z) \neq 0$.
3. $\forall\, x : \mathbb{R} \bullet \exp(x) > 0$.
4. exp *is strictly-increasing on* $\mathbb{R}$.[101]
5. $\forall\, n : \mathbb{N} \bullet x^{-n} \exp(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$.[102]

[99] I.e. $\forall\, z : \mathbb{C} \bullet \exp(z)$ is defined.
[100] I.e. exp is a homomorphism : $(\mathbb{C}, +) \to (\mathbb{C}^*, \cdot)$.
[101] I.e. $\mathbb{R} \lhd \exp$ is strictly-increasing.
[102] I.e. $x^n \in o(\exp(x))$.

6. $\forall\, n : \mathbb{N} \bullet x^n \exp(x) \longrightarrow 0$ *as* $x \longrightarrow -\infty$.
7. $\exp$ *restricts to a bijection* $: \mathbb{R} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{R}_+$.

Proof.

1. By Theorem 4.2.2, $\dfrac{\mathrm{d}}{\mathrm{d}X}\displaystyle\sum_{n=0}^{\infty}\frac{X^n}{n!} = \sum_{n=1}^{\infty}\frac{nX^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{X^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{X^n}{n!}$.
2. By Lemma 4.3.3, $\exp(z)\cdot\exp(-z) = \exp(0) = 1$, so $\exp(z) \neq 0$.
3. By Lemma 4.3.3, $\exp(x) = \exp\left(\dfrac{x}{2}\right)^2 \geqslant 0$, and by (2).
4. By (1) and (3), $\exp'(x) = \exp(x) > 0$, and by Corollary 3.2.11.
5. Given $m > 0$, let $M = m(n+1)!$, then

$$\forall\, x : \mathbb{R} \mid x > M \bullet x^{-n}\exp(x) > x^{-n}\frac{x^{n+1}}{(n+1)!} = \frac{x}{(n+1)!} > m$$

6. By Lemma 4.3.3 and (5),

$$\lim_{x\to-\infty}|x^n\exp(x)| = \lim_{x\to\infty}|x^n\exp(-x)| = \lim_{x\to\infty}\left|\frac{1}{x^{-n}\exp(x)}\right| = 0$$

7. By (4), (5), (6), and Theorem 3.2.12.

□

Definition 4.3.5. $\ln : \mathbb{R}_+ \to \mathbb{R}$ *is the inverse function of* $\exp$ *on* $\mathbb{R}$.[103]

Proposition 4.3.6.

1. $\forall\, x, y : \mathbb{R}_+ \bullet \ln(xy) = \ln x + \ln y$.[104]
2. $\ln$ *is differentiable, and* $\forall\, x : \mathbb{R}_+ \bullet \ln' x = \dfrac{1}{x}$.
3. $\ln 1 = 0$, $\forall\, x : (0,1) \bullet \ln x < 0$, *and* $\forall\, x : (1,\infty) \bullet \ln x > 0$.
4. $\forall\, n : \mathbb{N} \bullet \dfrac{\sqrt[n]{x}}{\ln x} \longrightarrow \infty$ *as* $x \longrightarrow \infty$.
5. $\forall\, n : \mathbb{N} \bullet \sqrt[n]{x}\ln x \longrightarrow 0$ *as* $x \longrightarrow 0$.

[103] I.e. $\ln = (\mathbb{R} \lhd \exp)^{-1}$.

[104] I.e. $\ln$ is a homomorphism $: (\mathbb{R}_+, \cdot) \to (\mathbb{R}, +)$.

Proof.

1. $\ln x + \ln y = \ln \exp(\ln x + \ln y) = \ln(\exp(\ln x) \cdot \exp(\ln y)) = \ln(xy)$.
2. By Theorem 3.2.12, ln is differentiable, and
$$\forall\, x : \mathbb{R}_+ \bullet \ln' x = (\exp^{-1})'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$
3. $\exp(0) = 1$, and ln is strictly-increasing.
4. By Lemma 2.1.6 and Proposition 4.3.4.5, letting $y = \ln x$,
$$\lim_{x\to\infty} \frac{\sqrt[n]{x}}{\ln x} = \lim_{y\to\infty} \sqrt[n]{\frac{\exp(y)}{y^n}} = \infty$$
5. By Lemma 2.1.6 and Proposition 4.3.4.6, letting $y = \ln x$,
$$\lim_{x\to 0} \sqrt[n]{x} \ln x = \lim_{y\to -\infty} \sqrt[n]{\exp(y) y^n} = 0$$

□

Theorem 4.3.7. $\ln(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n$ *is valid on* $(-1, 1)$.[105]

Proof. Define $f(X) = \ln(1+X)$. By Proposition 4.3.6.2, $f'(X) = \frac{1}{1+X}$, and by induction on n, for $n \geqslant 1$, $f^{(n)}(X) = \frac{(-1)^{n+1}(n-1)!}{(1+X)^n}$. Hence the given power series is the Taylor series of f. By Corollary 3.4.5, f equals its Taylor series at x iff $\frac{R_n(x)}{n!} x^n \longrightarrow 0$ as $n \longrightarrow \infty$.

By Theorem 3.4.4, $\frac{R_n(x)}{n!} x^n = \frac{(-1)^{n+1} x^n}{n(1+t_n x)^n}$ for some $(t_n)_{n:\mathbb{N}}$ in $(0, 1)$, which converges to 0 for $0 < x < 1$. Cauchy's form for the remainder gives $\frac{R_n(x)}{n!} x^n = \frac{(-1)^{n+1}(1-t_n)^{n-1} x^n}{(1+t_n x)^n} \longrightarrow 0$ for $-1 < x < 0$. □

[105]Since the radius of convergence is 1, we could define ln on $\mathbb{C}$ by this series and Proposition 4.3.6.1. However, exp is not injective on $\mathbb{C}$, so it doesn't have an inverse function in general.

4.4 Trigonometric and Hyperbolic Functions

Definition 4.4.1. *The functions* $\cos, \sin, \cosh, \sinh : \mathbb{C} \to \mathbb{C}$ *are given by*

1. $\cos z = \dfrac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}$, *and* $\sin z = \dfrac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}$.
2. $\cosh z = \dfrac{\exp(z) + \exp(-z)}{2}$, *and* $\sinh z = \dfrac{\exp(z) - \exp(-z)}{2}$.[106]

Proposition 4.4.2.

1. $\cos X = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n}$, *and* $\sin X = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1}$.
2. $\cosh X = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!} X^{2n}$, *and* $\sinh X = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} X^{2n+1}$.
3. $\cos$, $\sin$, $\cosh$, $\sinh$ *all restrict to functions* $: \mathbb{R} \to \mathbb{R}$.

Proof. By Definition 4.3.1, Proposition 4.1.4.2, and the fact that the terms in the power series are real. □

Proposition 4.4.3. $\forall\, z, w : \mathbb{C},\ a, b : \mathbb{R}$,

1. $\exp(z) = \cos z + \mathrm{i} \sin z$.
2. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.
3. $\sin(z + w) = \sin z \cos w + \cos z \sin w$.
4. $(\cos z)^2 + (\sin z)^2 = 1$.
5. $(\cosh z)^2 = 1 + (\sinh z)^2$.
6. $|\sin(a + \mathrm{i}b)|^2 = (\sin a)^2 + (\sinh b)^2$
7. $\cos' = -\sin$, $\sin' = \cos$, $\cosh' = \sinh'$, *and* $\sinh' = \cosh$.

Proof.

1. By Definition 4.4.1.1.

[106] I.e. $\cosh z = \cos(\mathrm{i}z)$, and $\mathrm{i} \sinh z = \sin(\mathrm{i}z)$.

2. $\cos z \cos w - \sin z \sin w$

$$= \frac{\exp(\mathrm{i}z)+\exp(-\mathrm{i}z)}{2} \cdot \frac{\exp(\mathrm{i}w)+\exp(-\mathrm{i}w)}{2} - \frac{\exp(\mathrm{i}z)-\exp(-\mathrm{i}z)}{2\mathrm{i}} \cdot \frac{\exp(\mathrm{i}w)-\exp(-\mathrm{i}w)}{2\mathrm{i}}$$

$$= \frac{\exp(\mathrm{i}z)\exp(\mathrm{i}w)+\exp(\mathrm{i}z)\exp(-\mathrm{i}w)+\exp(-\mathrm{i}z)\exp(\mathrm{i}w)+\exp(-\mathrm{i}z)\exp(\mathrm{i}-w)}{4}$$

$$+ \frac{\exp(\mathrm{i}z)\exp(\mathrm{i}w)-\exp(\mathrm{i}z)\exp(-\mathrm{i}w)-\exp(-\mathrm{i}z)\exp(\mathrm{i}-w)+\exp(-\mathrm{i}z)\exp(-\mathrm{i}w)}{4}$$

$$= \frac{2\exp(\mathrm{i}z)\exp(\mathrm{i}w)+2\exp(-\mathrm{i}z)\exp(\mathrm{i}-w)}{4} = \frac{\exp(\mathrm{i}(z+w))+\exp(-\mathrm{i}(z+w))}{2} = \cos(z+w).$$

3. Similar to (2).

4. By Definition 4.4.1.1, $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$.
   By (2), $1 = \cos 0 = \cos(z - z) = \cos z \cos(-z) - \sin z \sin(-z)$
   $= \cos z \cos z + \sin z \sin z.$

5. Similar to (4).

6. $$|\sin(a + \mathrm{i}b)|^2 = |\sin a \cos(\mathrm{i}b) + \cos a \sin(\mathrm{i}b)|^2$$
   $$= |\sin a \cosh b + \mathrm{i} \cos a \sinh b|^2$$
   $$= (\sin a \cosh b)^2 + (\cos a \sinh b)^2$$
   $$= (\sin a)^2(1 + (\sinh b)^2) + (1 - (\sin a)^2)(\sinh b)^2$$
   $$= (\sin a)^2 + (\sinh b)^2.$$

7. By Proposition 4.3.4.1 and Theorem 3.1.4.

□

Lemma 4.4.4.

1. $\forall\, z : \mathbb{C} \mid \sin z = 0 \bullet z \in \mathbb{R}$.

2. $\forall\, w : \mathbb{C} \bullet \sin w = 0 \Leftrightarrow \forall\, z : \mathbb{C} \bullet \exp(z) = \exp(z + 2\mathrm{i}w)$.[107]

Proof.

1. Let $z = a + \mathrm{i}b$ for $a, b : \mathbb{R}$. By Proposition 4.4.3.6, if $\sin z = 0$ then $\sin a = \sinh b = 0$. $\forall\, x : \mathbb{R} \bullet \sinh' x = \cosh x > 0$, so by Theorem 3.2.12, sinh is injective, and hence $\sinh b = 0$ iff $b = 0$.

2. $$\sin w = 0 \Leftrightarrow \exp(\mathrm{i}w) = \exp(-\mathrm{i}w)$$
   $$\Leftrightarrow \exp(2\mathrm{i}w) = 1$$
   $$\Leftrightarrow \exp(z) = \exp(z)\exp(2\mathrm{i}w) = \exp(z + 2\mathrm{i}w).$$

□

[107] I.e. $2\mathrm{i}w$ is a *period* of exp.

Theorem 4.4.5. $\nexists\, x : (0, 2] \bullet \sin x = 0$, *and* $\exists\, \pi : (2, 4) \bullet \sin \pi = 0$.

Proof. By Theorem 3.4.4, $\forall\, x : \mathbb{R}$,

- $\exists\, t : [0, 1] \bullet \sin x = x - \frac{1}{6}x^3 \cos(tx)$.
- $\exists\, t : [0, 1] \bullet \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cos(tx)$.

By Proposition 4.4.3.4, $|\cos(tx)| \leqslant 1$, so

- $\forall\, x : (0, 2] \bullet \sin x \geqslant x(1 - \frac{1}{6}x^2) > 0$.
- $\cos 1 \geqslant 1 - \frac{1}{2} - \frac{1}{24} > 0$.
- $\cos 2 \leqslant 1 - 2 + \frac{16}{24} < 0$.

Hence by Theorem 2.3.2, $\exists\, x : (1, 2) \bullet \cos x = 0$. By Proposition 4.4.3.3, $\sin(2x) = 2 \sin x \cos x = 0$. □

Definition 4.4.6. $\pi = \inf\{\, x : \mathbb{R}_+ \bullet \sin x = 0 \,\}$.

By Theorem 4.4.5, the infimum exists, and $2 < \pi < 4$. By Definition 4.4.1.1 and Lemma 4.4.4.2, cos and sin are periodic with period 2π.

Corollary 4.4.7. $\{\, z : \mathbb{C} \mid \sin z = 0 \,\} = \{\, n : \mathbb{Z} \bullet n\pi \,\}$.

Proof. sin is continuous, so $\sin \pi = 0$. By Proposition 4.4.3, $\cos \pi = \pm 1$ and so $\forall\, x : \mathbb{R} \bullet \sin(x + \pi) = \sin x \cos \pi + \cos x \sin \pi = \pm \sin x$, and inductively $\forall\, n : \mathbb{Z} \bullet \sin(n\pi) = 0$.

Conversely, given $z : \mathbb{C} \mid \sin z = 0$, by Lemma 4.4.4.1 $z \in \mathbb{R}$, so $\exists\, n : \mathbb{Z} \bullet 0 \leqslant z - n\pi < \pi$ and $\sin(z - n\pi) = 0$, so $z = n\pi$. □

4.5 Irrational Powers

Definition 4.5.1. $\mathrm{e} = \exp(1)$.

Lemma 4.5.2. $\forall\, x : \mathbb{Q} \bullet \exp(x) = \mathrm{e}^x$.

Proof.

1. $\mathrm{e}^0 = 1 = \exp(0)$.
2. By induction on $n : \mathbb{N}$, $\mathrm{e}^{n+1} = \mathrm{e}^n \mathrm{e} = \exp(n) \exp(1) = \exp(n + 1)$.
3. $\exp(-n) \exp(n) = \exp(0) = 1$, so $\exp(-n) = \frac{1}{\mathrm{e}^n} = \mathrm{e}^{-n}$.
4. For $\frac{p}{q} : \mathbb{Q}$, $(\mathrm{e}^{\frac{p}{q}})^q = \mathrm{e}^p = \exp(p)$ so $\mathrm{e}^{\frac{p}{q}} = \sqrt[q]{\exp(p)} = \sqrt[q]{\mathrm{e}^p}$.

□

Definition 4.5.3. *For* $a : \mathbb{R}_+, z : \mathbb{C}$*,* $a^z = \exp(z \ln a)$*.*

By Lemma 4.5.2, this is a direct generalisation of Definition 1.1.5.

Proposition 4.5.4. *For* $\alpha : \mathbb{C}$*,* $\dfrac{\mathrm{d}}{\mathrm{d}x} x^\alpha = \alpha x^{\alpha - 1}$*.*

Proof. By Theorem 3.1.4, $\dfrac{\mathrm{d}}{\mathrm{d}x} \exp(\alpha \ln x) = \dfrac{\alpha}{x} x^\alpha$. □

Definition 4.5.5. *For* $\alpha : \mathbb{C}, n : \mathbb{N}$*,* $\displaystyle\binom{\alpha}{n} = \frac{1}{n!} \prod_{k=0}^{n-1} (\alpha - k)$*.*

Theorem 4.5.6 (Newton's Generalised Binomial Theorem)**.**

$$\forall\, \alpha : \mathbb{C} \bullet (1 + X)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} X^n \textit{ is valid on } (-1, 1).$$

Proof. Define $f(X) = (1 + X)^\alpha$. Inductively, $f^{(n)}(X) = n! \dbinom{\alpha}{n} (1 + X)^{\alpha - n}$, so $f^{(n)}(0) = n! \dbinom{\alpha}{n}$. By Corollary 3.4.5, the power series is valid at $x : \mathbb{R}$ iff $\dfrac{R_n(x)}{n!} x^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $k : \mathbb{N} \mid k \geqslant |\alpha|$. For $|x| < 1$, $\left| \dbinom{\alpha}{n} x^n \right| \leqslant \left| \alpha^k (n + k)^k x^n \right| \longrightarrow 0$.

For $0 < x < 1$, $n > k$, Lagrange's form for the remainder gives

$$\left| \frac{R_n(x)}{n!} x^n \right| = \left| \binom{\alpha}{n} (1 + t_n x)^{\alpha - n} x^n \right| < \left| \binom{\alpha}{n} x^n \right| \longrightarrow 0$$

for some $(t_n)_{n:\mathbb{N}}$ in $(0, 1)$.

For $-1 < x < 0$, $n > k$, Cauchy's form for the remainder gives

$$\begin{aligned} \left| \frac{R_n(x)}{n!} x^n \right| &= \left| n (1 - t_n)^{n-1} \binom{\alpha}{n} (1 + t_n x)^{\alpha - n} x^n \right| \\ &= \left| (1 + t_n x)^{\alpha - 1} x \alpha \binom{\alpha - 1}{n - 1} \left(\frac{1 - t_n}{1 + t_n x} \right)^{n-1} x^{n-1} \right| \\ &\leqslant \left| 2^k x \alpha \binom{\alpha - 1}{n - 1} x^{n-1} \right| \longrightarrow 0. \end{aligned}$$

□

5 Integration

5.1 Riemann Integration

We will follow the definition of integration given by Riemann.

Definition 5.1.1.

1. *$\mathscr{D} : \mathbb{P}[a, b]$ is a* ***dissection*** *of $[a, b]$ if $\mathscr{D}$ is finite, and $a, b \in \mathscr{D}$.*
2. *$\mathrm{D}[a, b]$ is the set of all dissections of $[a, b]$.*

Definition 5.1.2. *If $f : [a, b] \to \mathbb{R}$ is bounded, and $\mathscr{D} = \{x_0, \dots x_n\}$ is a dissection of $[a, b]$ enumerated in order,*[108]

1. *$\mathrm{U}(f, \mathscr{D}) = \displaystyle\sum_{k=0}^{n-1}(x_{k+1} - x_k) \sup_{x:[x_k, x_{k+1}]} f(x)$ is the* ***upper sum*** *of f over $\mathscr{D}$.*
2. *$\mathrm{L}(f, \mathscr{D}) = \displaystyle\sum_{k=0}^{n-1}(x_{k+1} - x_k) \inf_{x:[x_k, x_{k+1}]} f(x)$ is the* ***lower sum*** *of f over $\mathscr{D}$.*

Trivially, $\forall\, \mathscr{D} : \mathrm{D}[a, b] \bullet \mathrm{L}(f, \mathscr{D}) \leqslant \mathrm{U}(f, \mathscr{D})$.

Lemma 5.1.3. *If $f : [a, b] \to \mathbb{R}$ is bounded, and $\mathscr{D}_1, \mathscr{D}_2 : \mathrm{D}[a, b]$,*

1. *If $\mathscr{D}_1 \subseteq \mathscr{D}_2$, then $\mathrm{L}(f, \mathscr{D}_1) \leqslant \mathrm{L}(f, \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_1)$.*
2. *$\mathrm{L}(f, \mathscr{D}_1) \leqslant \mathrm{U}(f, \mathscr{D}_2)$.*

Proof.

1. $\mathscr{D}_2 = \mathscr{D}_1$ is trivial. By induction on $\#\mathscr{D}_2 - \#\mathscr{D}_1$, we need only consider the case where $\mathscr{D}_2 = \mathscr{D}_1 \cup \{x'\}$, with $x_k < x' < x_{k+1}$.

$$\begin{aligned}
&(x_{k+1} - x_k) \sup_{x:[x_k, x_{k+1}]} f(x) \\
&\quad = \left[(x' - x_k) \sup_{x:[x_k, x_{k+1}]} f(x)\right] + \left[(x_{k+1} - x') \sup_{x:[x_k, x_{k+1}]} f(x)\right] \\
&\qquad \geqslant \left[(x' - x_k) \sup_{x:[x_k, x']} f(x)\right] + \left[(x_{k+1} - x') \sup_{x:[x', x_{k+1}]} f(x)\right]
\end{aligned}$$

and hence $\mathrm{U}(f, \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_1)$. L is similar.

2. By (1), $\mathrm{L}(f, \mathscr{D}_1) \leqslant \mathrm{L}(f, \mathscr{D}_1 \cup \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_1 \cup \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_2)$.

□

[108] I.e. $x_0 = a$, $x_n = b$, and $x_i < x_j$ iff $i < j$.

Definition 5.1.4. *If $f : [a,b] \to \mathbb{R}$ is bounded,*

1. $\underline{\int_a^b} f = \sup_{\mathscr{D}:\mathrm{D}[a,b]} \mathrm{L}(f, \mathscr{D})$ *is the* ***lower integral*** *of f.*

2. $\overline{\int_a^b} f = \inf_{\mathscr{D}:\mathrm{D}[a,b]} \mathrm{U}(f, \mathscr{D})$ *is the* ***upper integral*** *of f.*

3. *If* $\underline{\int_a^b} f = \overline{\int_a^b} f$, *then f is* ***Riemann-integrable****, and* $\int_a^b f = \overline{\int_a^b} f$.

By Lemma 5.1.3.2, every upper sum is an upper bound of the set of lower sums, and every lower sum is a lower bound of the set of upper sums. It follows immediately that the lower and upper integrals always exist, and

$$\underline{\int_a^b} f \leqslant \overline{\int_a^b} f$$

We will write “integrable” to mean “Riemann-integrable”. Where f is defined by an expression, we will write e.g. $\int_a^b f \, \mathrm{d}x$ to clarify that f is a function of x.

Proposition 5.1.5. *If $f : [a,b] \to \mathbb{R}$ is bounded, then f is integrable iff $\forall \epsilon : \mathbb{R}_+ \bullet \exists \mathscr{D} : \mathrm{D}[a,b] \bullet \mathrm{U}(f, \mathscr{D}) - \mathrm{L}(f, \mathscr{D}) < \epsilon$.*

Proof. $\forall \epsilon : \mathbb{R}_+ \bullet \exists \mathscr{D} : \mathrm{D}[a,b] \bullet 0 \leqslant \overline{\int_a^b} f - \underline{\int_a^b} f \leqslant \mathrm{U}(f, \mathscr{D}) - \mathrm{L}(f, \mathscr{D}) < \epsilon$, so the upper and lower integrals are equal.

Conversely, given $\epsilon > 0$, by Proposition 1.3.6,

- $\exists \mathscr{D}_1 : \mathrm{D}[a,b] \bullet \mathrm{L}(f, \mathscr{D}_1) \geqslant \underline{\int_a^b} f - \frac{\epsilon}{2} = \int_a^b f - \frac{\epsilon}{2}$.

- $\exists \mathscr{D}_2 : \mathrm{D}[a,b] \bullet \mathrm{U}(f, \mathscr{D}_2) \leqslant \overline{\int_a^b} f + \frac{\epsilon}{2} = \int_a^b f + \frac{\epsilon}{2}$.

and so by Lemma 5.1.3.1,

$$\mathrm{U}(f, \mathscr{D}_1 \cup \mathscr{D}_2) - \mathrm{L}(f, \mathscr{D}_1 \cup \mathscr{D}_2) \leqslant \mathrm{U}(f, \mathscr{D}_2) - \mathrm{L}(f, \mathscr{D}_1) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

5.2 Properties of Integrals

Proposition 5.2.1. *For $f, g : [a,b] \to \mathbb{R}$ integrable,*

1. *If* $\forall\, x : [a,b] \bullet f(x) \leqslant g(x)$, *then* $\displaystyle\int_a^b f \leqslant \int_a^b g$.
2. $\forall\, \lambda, \mu : \mathbb{R} \bullet \lambda f + \mu g$ *is integrable, and* $\displaystyle\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g$.[109]
3. $\max\{f(x), g(x)\} : [a,b] \to \mathbb{R}$ *is integrable.*
4. $|f|$ *is integrable, and* $\displaystyle\left|\int_a^b f\right| \leqslant \int_a^b |f|$.
5. $f \cdot g$ *is integrable.*
6. *For* $h : [a,b] \to \mathbb{R}$, *if* $\{\, x : [a,b] \mid f(x) \neq h(x) \,\}$ *is finite,*[110] *then* h *is integrable, and* $\displaystyle\int_a^b h = \int_a^b f$.

Proof.

1. $\forall\, \mathscr{D} : \mathrm{D}[a,b] \bullet \displaystyle\overline{\int_a^b} f \leqslant \mathrm{U}(f, \mathscr{D}) \leqslant \mathrm{U}(g, \mathscr{D})$, so $\displaystyle\overline{\int_a^b} f \leqslant \overline{\int_a^b} g$.
2. $\forall\, \mathscr{D} : \mathrm{D}[a,b]$,

   - $\mathrm{U}(-f, \mathscr{D}) = -\,\mathrm{L}(f, \mathscr{D})$, so $\displaystyle\overline{\int_a^b} (-f) = -\underline{\int_a^b} f$.
   - $\mathrm{L}(-f, \mathscr{D}) = -\,\mathrm{U}(f, \mathscr{D})$, so $\displaystyle\underline{\int_a^b} (-f) = -\overline{\int_a^b} f$.

   Hence, $-f$ is integrable, and $\displaystyle\int_a^b (-f) = -\int_a^b f$, so wlog $\lambda, \mu \geqslant 0$.

   Given $x_k, x_{k+1} : \mathscr{D}$,

   $$\sup_{x:[x_k,x_{k+1}]} (\lambda f + \mu g)(x) \leqslant \lambda \left[\sup_{x:[x_k,x_{k+1}]} f(x)\right] + \mu \left[\sup_{x:[x_k,x_{k+1}]} g(x)\right]$$

---

[109] I.e. $\displaystyle\int_a^b$ is a linear operator on integrable functions.

[110] Other definitions of integration may give similar but stronger results.

so $\forall\, \mathscr{D}_1, \mathscr{D}_2 : \mathrm{D}[a,b]$,

$$\begin{aligned}\overline{\int_a^b}(\lambda f + \mu\, g) &\leqslant \mathrm{U}(\lambda f + \mu\, g, \mathscr{D}_1 \cup \mathscr{D}_2)\\ &\leqslant \lambda\, \mathrm{U}(f, \mathscr{D}_1 \cup \mathscr{D}_2) + \mu\, \mathrm{U}(g, \mathscr{D}_1 \cup \mathscr{D}_2) \leqslant \lambda\, \mathrm{U}(f, \mathscr{D}_1) + \mu\, \mathrm{U}(g, \mathscr{D}_2)\end{aligned}$$

so $\overline{\int_a^b}(\lambda f + \mu\, g) \leqslant \lambda \overline{\int_a^b} f + \mu \overline{\int_a^b} g$.

Similarly, $\underline{\int_a^b}(\lambda f + \mu\, g) \geqslant \lambda \underline{\int_a^b} f + \mu \underline{\int_a^b} g$. Therefore,

$$\overline{\int_a^b}(\lambda f + \mu\, g) \leqslant \lambda \int_a^b f + \mu \int_a^b g \leqslant \underline{\int_a^b}(\lambda f + \mu\, g)$$

and the result follows.

3. $\max\{f(x), g(x)\} = g(x) + \max\{f(x) - g(x), 0\}$, so by (2), it is sufficient to show the result for $f_+(x) = \max\{f(x), 0\}$.

   Given $x_k, x_{k+1} : \mathscr{D} : \mathrm{D}[a,b]$,

$$\left[\sup_{x:[x_k,x_{k+1}]} f_+(x)\right] - \left[\inf_{x:[x_k,x_{k+1}]} f_+(x)\right] \leqslant \left[\sup_{x:[x_k,x_{k+1}]} f(x)\right] - \left[\inf_{x:[x_k,x_{k+1}]} f(x)\right]$$

   Hence by Proposition 5.1.5, given $\epsilon > 0$,

$$\exists\, \mathscr{D} : \mathrm{D}[a,b] \bullet \mathrm{U}(f_+, \mathscr{D}) - \mathrm{L}(f_+, \mathscr{D}) \leqslant \mathrm{U}(f, \mathscr{D}) - \mathrm{L}(f, \mathscr{D}) < \epsilon$$

   so f_+ is integrable.

4. By (2) and (3), $f_- = f - f_+$ is integrable, so $|f| = f_+ - f_-$ is integrable. $\forall\, x : [a,b] \bullet -|f(x)| \leqslant f(x) \leqslant |f(x)|$, so by (1),

$$-\int_a^b |f| \leqslant \int_a^b f \leqslant \int_a^b |f|$$

5. Given $x_k, x_{k+1} : \mathscr{D} : \mathrm{D}[a,b]$, let

   - $S = \sup_{x:[x_k,x_{k+1}]} |f(x)|$, and $S' = \sup_{x:[x_k,x_{k+1}]} |f(x)|^2$.
   - $I = \inf_{x:[x_k,x_{k+1}]} |f(x)|$, and $I' = \inf_{x:[x_k,x_{k+1}]} |f(x)|^2$.

with $S' = S^2$ and $I' = I^2$. $\exists\, M : \mathbb{R} \bullet \forall\, x : [a, b] \bullet |f(x)| < M$, so $S' - I' = (S + I)(S - I) < 2M(S - I)$.

By Proposition 5.1.5 and (4), given $\epsilon > 0$,

$$\exists\, \mathscr{D} : \mathrm{D}[a, b] \bullet \mathrm{U}(|f|, \mathscr{D}) - \mathrm{U}(|f|, \mathscr{D}) < \frac{\epsilon}{2M}$$

so $\mathrm{U}(|f|^2, \mathscr{D}) - \mathrm{L}(|f|^2, \mathscr{D}) < 2M\big(\mathrm{U}(|f|, \mathscr{D}) - \mathrm{U}(|f|, \mathscr{D})\big) < \epsilon$, and hence $|f|^2$ is integrable.

$f \cdot g = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2 = \frac{1}{4}|f + g|^2 - \frac{1}{4}|f - g|^2$, so by (2), $f \cdot g$ is integrable.

6. By (2), h is integrable iff $h - f$ is integrable, so it is sufficient to show the result for $f = 0$. Let $X = \{\, x : [a, b] \mid h(x) \neq 0 \,\}$, and $n = \#X$.

   $\forall\, \mathscr{D} : \mathrm{D}[a, b] \bullet \mathrm{L}(h, \mathscr{D}) = 0$, so $\underline{\int_a^b} h = 0$.

   Given $\epsilon > 0$, let $l_\epsilon(x) = \max\left\{a, x - \dfrac{\epsilon}{3n}\right\}$, $r_\epsilon(x) = \min\left\{b, x + \dfrac{\epsilon}{3n}\right\}$, and $\mathscr{D}_\epsilon = \{a, b\} \cup l_\epsilon(X) \cup r_\epsilon(X)$. Then $\mathrm{U}(h, \mathscr{D}_\epsilon) \leqslant n\left(2\dfrac{\epsilon}{3n}\right) < \epsilon$.

   Therefore $\overline{\int_a^b} h = 0$, and the result follows.

□

Theorem 5.2.2. *For $c : (a, b)$, $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff it is integrable on both $[a, c]$ and $[c, b]$, in which case* $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. $\forall\, \mathscr{D}_1 : \mathrm{D}[a, c]$, $\mathscr{D}_2 : \mathrm{D}[c, b] \bullet \mathscr{D}_1 \cup \mathscr{D}_2$ is a dissection of $[a, b]$, and

$$\overline{\int_a^b} f \leqslant \mathrm{U}(f, \mathscr{D}_1 \cup \mathscr{D}_2) = \mathrm{U}(f, \mathscr{D}_1) + \mathrm{U}(f, \mathscr{D}_2)$$

hence $\displaystyle\overline{\int_a^b} f \leqslant \overline{\int_a^c} f + \overline{\int_c^b} f$. Similarly for lower integrals.

Conversely, $\forall\, \mathscr{D} : \mathrm{D}[a, b]$, let $\mathscr{D}' = \mathscr{D} \cup \{c\}$, $\mathscr{D}_1 = \mathscr{D}' \cap [a, c]$, and $\mathscr{D}_2 = \mathscr{D}' \cap [c, b]$. Then $\mathscr{D}_1, \mathscr{D}_2$ are dissections of $[a, c]$, $[c, b]$ respectively, and

$$\overline{\int_a^c} f + \overline{\int_c^b} f \leqslant \mathrm{U}(f, \mathscr{D}_1) + \mathrm{U}(f, \mathscr{D}_2) = \mathrm{U}(f, \mathscr{D}') \leqslant \mathrm{U}(f, \mathscr{D})$$

hence $\displaystyle\overline{\int_a^c} f + \overline{\int_c^b} f \leqslant \overline{\int_a^b} f$. Similarly for lower integrals. □

5.3 Integrable Functions

Theorem 5.3.1. *If $f : [a,b] \to \mathbb{R}$ is monotonic, then it is integrable.*

Proof. Wlog f is non-decreasing. For $n : \mathbb{N}_+, k : \mathbb{N} \mid 0 \leqslant k \leqslant n$, let $x_{k,n} = a + \frac{b-a}{n}k$, and $\mathscr{D}_n = \{\, k : \mathbb{N} \mid 0 \leqslant k \leqslant n \bullet x_{k,n} \,\} \in \mathrm{D}[a,b]$. $\forall\, n : \mathbb{N}_+$,

- $\mathrm{U}(f, \mathscr{D}_n) = \dfrac{b-a}{n} \displaystyle\sum_{k=1}^{n} f(x_{k,n})$, and $\mathrm{L}(f, \mathscr{D}_n) = \dfrac{b-a}{n} \displaystyle\sum_{k=0}^{n-1} f(x_{k,n})$.
- $\mathrm{U}(f, \mathscr{D}_n) - \mathrm{L}(f, \mathscr{D}_n) = \dfrac{b-a}{n}\big(f(x_{n,n}) - f(x_{0,n})\big) = \dfrac{b-a}{n}\big(f(b) - f(a)\big)$.

hence $\mathrm{U}(f, \mathscr{D}_n) - \mathrm{L}(f, \mathscr{D}_n) \longrightarrow 0$ as $n \longrightarrow \infty$, and by Proposition 5.1.5, f is integrable. □

Corollary 5.3.2. *If $f : [a,b] \to \mathbb{R}$ is piecewise-monotonic,*[111] *then it is integrable.*

Proof. By Theorem 5.2.2 and Theorem 5.3.1. □

Theorem 5.3.3. *If $f : [a,b] \to \mathbb{R}$ is continuous, then it is integrable.*

Proof. Suppose f is not integrable. By Proposition 5.1.5, $\exists\, \epsilon : \mathbb{R}_+ \bullet \forall\, \mathscr{D} : \mathrm{D}[a,b] \bullet \mathrm{U}(f, \mathscr{D}) - \mathrm{L}(f, \mathscr{D}) \geqslant \epsilon$. Define $x_{n,k}$, $\mathscr{D}_n$ as in the proof of Theorem 5.3.1, and let $I_{k,n} = [x_{k,n}, x_{k+1,n}]$. Then

$$\forall\, n : \mathbb{N}_+ \bullet \exists\, k : \mathbb{N} \mid 0 \leqslant k < n \bullet \frac{\epsilon}{b-a} \leqslant \left[\sup_{x : I_{k,n}} f(x)\right] - \left[\inf_{x : I_{k,n}} f(x)\right]$$

so by Theorem 2.3.5,

$$\exists\, p_n, q_n : I_{k,n} \bullet f(q_n) - f(p_n) \geqslant \frac{\epsilon}{b-a}, \text{ but } |q_n - p_n| \leqslant \frac{b-a}{n}$$

By Theorem 1.5.7, $(p_n)_{n:\mathbb{N}}$ has a subsequence $(p_{m(n)})_{n:\mathbb{N}}$ converging to some $t : [a,b]$. $q_{m(n)} - p_{m(n)} \longrightarrow 0$, so $q_{m(n)} \longrightarrow t$. f is continuous at t, so by Theorem 2.1.2, $f(p_{m(n)}) \longrightarrow f(t)$ and $f(q_{m(n)}) \longrightarrow f(t)$, contradicting $\forall\, n : \mathbb{N}_+ \bullet f(q_{m(n)}) - f(p_{m(n)}) \geqslant \dfrac{\epsilon}{b-a}$. Hence f is integrable. □

[111] I.e. $\exists\, \mathscr{D} : \mathrm{D}[a,b] \bullet f$ is monotonic on each $[x_k, x_{k+1}]$.

5.4 Calculus

Definition 5.4.1. *If* $f : [a, b] \to \mathbb{R}$ *is integrable, then* $\displaystyle\int_b^a f = -\int_a^b f$.[112]

Lemma 5.4.2. *If* $f : [a, b] \to \mathbb{R}$ *is integrable, then* $F : [a, b] \to \mathbb{R}$ *given by* $F(x) = \displaystyle\int_a^x f$ *is continuous.*

Proof. By Definition 5.1.4, $\exists\, m : \mathbb{R} \bullet \forall\, x : [a, b] \bullet |f(x)| \leqslant m$. By Theorem 5.2.2, $\forall\, h : \mathbb{R} \mid x + h \in [a, b] \bullet F(x+h) = F(x) + \displaystyle\int_x^{x+h} f$, so by Proposition 5.2.1.1, $|F(x+h) - F(x)| = \left|\displaystyle\int_x^{x+h} f\right| \leqslant m\,|h| \longrightarrow 0$ as $h \longrightarrow 0$. Hence F is continuous at x. □

Theorem 5.4.3 (Fundamental Theorem of Calculus)**.** *If* $f : [a, b] \to \mathbb{R}$ *is integrable, and continuous at* $c : (a, b)$*, then* $F : [a, b] \to \mathbb{R}$ *given by* $F(x) = \displaystyle\int_a^x f$ *is differentiable at* c*, and* $F'(c) = f(c)$.[113]

Proof. Given $\epsilon > 0$, $\exists\, \delta : \mathbb{R}_+ \bullet \forall\, h : (-\delta, \delta) \bullet f(c) - \epsilon < f(c+h) < f(c) + \epsilon$, so

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{1}{h}\right| \left|\int_c^{c+h} f - hf(c)\right|$$
$$= \frac{1}{|h|} \left|\int_c^{c+h} (f - f(c))\right| < \frac{1}{|h|}\,|h\epsilon| = \epsilon$$

The result follows by Proposition 3.1.5. □

[112] A generalisation of Theorem 5.2.2 immediately follows: if $f : [a, b] \to \mathbb{R}$ is integrable, then $\forall\, p, q, r : [a, b] \bullet \displaystyle\int_p^r f = \int_p^q f + \int_q^r f$.

[113] I.e. $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is a *left-inverse* of $\displaystyle\int_a^x$ on continuous functions.

Corollary 5.4.4. *For $f, \phi : [a, b] \to \mathbb{R}$ continuous, if f is continuous and ϕ is differentiable with $\phi' = f$, then* $\int_a^b f = \phi(b) - \phi(a)$.

Proof. Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. By Theorem 5.4.3, $F - \phi$ is differentiable, and $(F - \phi)' = 0$. By Theorem 3.2.6,

$$\forall\, x : [a, b] \bullet (F - \phi)(x) = (F - \phi)(a) = -\phi(a)$$

so $F = \phi - \phi(a)$, and hence $\int_a^b f = F(b) = \phi(b) - \phi(a)$. □

Theorem 5.4.5 (Integration by Parts)**.** *For $f, g, F, G : [a, b] \to \mathbb{R}$ with f, g continuous, F, G differentiable, and $F' = f$, $G' = g$,*

$$\int_a^b F \cdot g = \big(F(b)G(b) - F(a)G(a)\big) - \int_a^b f \cdot G$$

Proof. By Proposition 3.1.3.2, $(F \cdot G)' = F' \cdot G + F \cdot G' = f \cdot G + F \cdot g$ is continuous, so by Corollary 5.4.4,

$$\int_a^b F \cdot g = \int_a^b (F \cdot G)' - \int_a^b f \cdot G = \big((F \cdot G)(b) - (F \cdot G)(a)\big) - \int_a^b f \cdot G$$

□

Theorem 5.4.6 (Integration by Substitution)**.** *For $f : [a, b] \to \mathbb{R}$ continuously differentiable,*[114] *and $g : f([a, b]) \to \mathbb{R}$ continuous,*

$$\int_{f(a)}^{f(b)} g = \int_a^b (g \circ f) \cdot f'$$

Proof. By Corollary 2.3.6, $f([a, b]) = [c, d]$ for some $c, d : \mathbb{R}$.

Define $G : [c, d] \to \mathbb{R}$, $h : [a, b] \to \mathbb{R}$ by $G(x) = \int_c^x g$, $h(x) = G(f(x))$. By Theorem 3.1.4, h is differentiable, and $h' = (g \circ f) \cdot f'$ is continuous. Therefore by Theorem 5.4.3 and Corollary 5.4.4,

$$\int_a^b (g \circ f) \cdot f' = \int_a^b h' = h(b) - h(a) = G(f(b)) - G(f(a)) = \int_{f(a)}^{f(b)} G' = \int_{f(a)}^{f(b)} g$$

□

[114] I.e. f is differentiable, and f' is continuous.

5.5 The Mean Value Theorem for Integrals

Theorem 5.5.1 (Mean Value Theorem for Integrals)**.** *For* $f, g : [a, b] \to \mathbb{R}$ *integrable,*

1. *If* $\exists\, l, u : \mathbb{R} \bullet \forall\, x : [a, b] \bullet l \leqslant f(x) \leqslant u$ *and* $g(x) \geqslant 0$*, then* $\exists\, m : [l, u] \bullet \int_a^b f \cdot g = m \int_a^b g$.
2. *If* f *is continuous,* $\exists\, c : [a, b] \bullet \int_a^b f \cdot g = f(c) \int_a^b g$.

Proof.

1. By Proposition 5.2.1.1, $l \int_a^b g \leqslant \int_a^b f \cdot g \leqslant u \int_a^b g$.
2. By Theorem 2.3.5 and (1), $\exists f(c) : f([a, b])$.

□

Theorem 5.5.2. *For* $f : [a, a + \delta) \to \mathbb{R}$ *n-times continuously differentiable,*[115]

$$\forall\, h : [0, \delta) \bullet f(a + h) = \left[\sum_{k=0}^{n-1} \frac{f^{(k)}}{k!} h^k\right] + \frac{R_n(h)}{n!} h^n$$

where $R_n(h) = n \int_0^1 (1 - t)^{n-1} f^{(n)}(a + th)\,\mathrm{d}t$.

Proof. By Theorem 5.4.6, substituting $x = th$,

$$\frac{h^n}{(n-1)!} \int_0^1 (1 - t)^{n-1} f^{(n)}(a + th)\,\mathrm{d}t = \frac{1}{(n-1)!} \int_0^h (h - x)^{n-1} f^{(n)}(a + x)\,\mathrm{d}x$$

Inductively applying Theorem 5.4.5,

$$\begin{aligned}
&\frac{1}{(n-1)!} \int_0^h (h - x)^{n-1} f^{(n)}(a + x)\,\mathrm{d}x \\
&\quad = -\frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{1}{(n-2)!} \int_0^h (h - x)^{n-2} f^{(n-1)}(a + x)\,\mathrm{d}x \\
&\quad\quad = \left[-\sum_{k=1}^{n-1} \frac{f^{(k)}}{k!} h^k\right] + \int_0^h f'(a + x)\,\mathrm{d}x
\end{aligned}$$

[115] I.e. f is n-times-differentiable, and $f^{(n)}$ is continuous.

and by Corollary 5.4.4, $\int_0^h f'(a+x)\,\mathrm{d}x = f(a+h) - f(a)$. The result follows. □

Corollary 5.5.3.

1. $\exists\, t : [0,1] \bullet R_n(h) = f^{(n)}(a+th)$.
2. $\exists\, t : [0,1] \bullet R_n(h) = n(1-t)^{n-1} f^{(n)}(a+th)$.

Proof. By Theorem 5.5.1.2, with

1. $f(t) = f^{(n)}(a+th)$, $g(t) = n(1-t)^{n-1}$.
2. $f(t) = n(1-t)^{n-1} f^{(n)}(a+th)$, $g(t) = 1$.

□

5.6 Improper Integrals

Definition 5.6.1. *For $f : \mathbb{R} \to \mathbb{R}$, where the limits exist,*

1. *If $\forall\, x : [a,b) \bullet f$ is integrable on $[a,x]$, then $\int_a^b f = \lim_{x \to b^-} \int_a^x f$.*
2. *If $\forall\, x : (a,b] \bullet f$ is integrable on $[x,b]$, then $\int_a^b f = \lim_{x \to a^+} \int_x^b f$.*
3. *If $\forall\, x : [a,\infty) \bullet f$ is integrable on $[a,x]$, then $\int_a^\infty f = \lim_{x \to \infty} \int_a^x f$.*
4. *If $\forall\, x : (-\infty,b] \bullet f$ is integrable on $[x,b]$, then $\int_{-\infty}^b f = \lim_{x \to -\infty} \int_x^b f$.*

and f is ***improperly-integrable*** *on the given interval.*

Where f is integrable on $[a,b]$, by Lemma 5.4.2, (1) and (2) coincide with Definition 5.1.4.3. By Theorem 5.2.2, we can extend these definitions to *piecewise*-improperly-integrable[116] functions; e.g.

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f$$

independent of the choice of $c : \mathbb{R}$.

[116] I.e. f is improperly-integrable on each of a set of intervals whose union is $\operatorname{dom} f$.

Lemma 5.6.2. *For* $f : [a, \infty) \to \mathbb{R}$*, if* f *is non-negative,*[117] *and* $\forall\, x : [a, \infty) \bullet f$ *is integrable on* $[a, x]$*, then* $\displaystyle\int_a^\infty f$ *exists iff* $\displaystyle\left(\int_a^n f\right)_{n:\mathbb{N}}$ *is bounded.*

Proof. Define $F : [a, \infty)$ by $F(x) = \displaystyle\int_a^x f$. F is non-decreasing, so $\displaystyle\left(\int_a^n f\right)_{n:\mathbb{N}}$ is bounded iff F is bounded.

Let $l = \sup_{x:[a,\infty)} F(x)$. Given $\epsilon > 0$, $\exists\, M : [a, \infty) \bullet F(M) > l - \epsilon$, so $\forall\, x : [a, \infty) \mid x \geqslant M \bullet l - \epsilon < F(x) \leqslant l$. Therefore, $F(x) \longrightarrow l$ as $x \longrightarrow \infty$.

The converse is trivial. □

Theorem 5.6.3 (Maclaurin's Integral Test)**.** *For* $f : [0, \infty) \to \mathbb{R}$ *non-negative and non-increasing,*

1. $\displaystyle\sum_{n=0}^{\infty} f(n)$ *converges iff* $\displaystyle\int_0^\infty f$ *converges.*

2. $\displaystyle\exists\, l : [0, f(0)] \bullet \left[\sum_{k=0}^{n} f(k)\right] - \left[\int_0^n f\right] \longrightarrow l.$

Proof. By Theorem 5.3.1, $\forall\, x : [0, \infty) \bullet f$ is integrable on $[0, x]$.

1. By Proposition 5.2.1.1, $\forall\, n : \mathbb{N} \bullet f(n) \geqslant \displaystyle\int_n^{n+1} f \geqslant f(n+1)$, so

$$\sum_{k=0}^{n-1} f(k) \geqslant \int_0^n f \geqslant \sum_{k=1}^{n} f(k)$$

   - If $\displaystyle\sum_{n=0}^{\infty} f(n)$ converges, then $\displaystyle\left(\int_0^n f\right)_{n:\mathbb{N}}$ is bounded-above by it.
   - If $\displaystyle\sum_{n=0}^{\infty} f(n)$ diverges, then $\displaystyle\left(\int_0^n f\right)_{n:\mathbb{N}}$ is unbounded-above.

   The result follows by Lemma 5.6.2.

2. For $n : \mathbb{N}$, define $a_n = \displaystyle\left[\sum_{k=0}^{n} f(k)\right] - \left[\int_0^n f\right]$.

[117] I.e. $\forall\, x : [a, \infty) \bullet f(x) \geqslant 0$.

$\forall\, n : \mathbb{N} \bullet 0 \leqslant f(n) \leqslant a_n \leqslant f(0)$, and by Proposition 5.2.1.1,

$$a_n - a_{n+1} = \left[\int_n^{n+1} f\right] - f(n+1) \geqslant 0$$

so $(a_n)_{n:\mathbb{N}}$ is non-increasing. By Theorem 1.4.2, $(a_n)_{n:\mathbb{N}}$ converges.

□

A Appendix

A.1 Real Intervals

For $a, b : \mathbb{R} \mid a < b$,

$$\begin{array}{c} [a \\ (a \\ (-\infty \end{array} , \begin{array}{c} b] \\ b) \\ \infty) \end{array} = \{x : \mathbb{R} \mid \begin{array}{c} (x \geqslant a) \\ (x > a) \\ \top \end{array} \wedge \begin{array}{c} (x \leqslant b) \\ (x < b) \\ \top \end{array} \}$$

- $\emptyset$, $[a, a] = \{a\}$, $[a, b]$, $[a, \infty)$, $(-\infty, b]$, and $(-\infty, \infty) = \mathbb{R}$ are **closed**.
- $\emptyset$, (a, b), (a, ∞), $(-\infty, b)$, and $(-\infty, \infty) = \mathbb{R}$ are **open**.
- $[a, b)$ and $(a, b]$ are "half-open".

Definition A.1.1. $\mathbb{R}_+ = (0, \infty)$.

A.2 Function Limits

For $S : \mathbb{P}\mathbb{R}$, $\Sigma : \mathbb{P}\mathbb{C}$, $f : S \to \mathbb{R}$, $g : \Sigma \to \mathbb{C}$, $a, l : \mathbb{R}$, and $\alpha, \lambda : \mathbb{C}$,

$f(x) \longrightarrow l$		$\epsilon : \mathbb{R}_+$					$\lvert f(x) - l\rvert < \epsilon$
$f(x) \longrightarrow \infty$		$m : \mathbb{R}$					$f(x) > m$
$f(x) \longrightarrow -\infty$		$m : \mathbb{R}$					$f(x) < m$
as	if $\forall$		$\bullet\ \exists$		$\bullet\ \forall\, x : S \mid$	$\bullet$	
$x \longrightarrow a$				$\delta : \mathbb{R}_+$	$0 < \lvert x - a\rvert < \delta$		
$x \longrightarrow a^+$				$\delta : \mathbb{R}_+$	$a < x < a + \delta$		
$x \longrightarrow a^-$				$\delta : \mathbb{R}_+$	$a - \delta < x < a$		
$x \longrightarrow \infty$				$M : \mathbb{R}$	$x \geqslant M$		
$x \longrightarrow -\infty$				$M : \mathbb{R}$	$x \leqslant M$		

$g(z) \longrightarrow \lambda$		$\epsilon : \mathbb{R}_+$					$\lvert g(z) - \lambda\rvert < \epsilon$
$g(z) \longrightarrow \infty$		$m : \mathbb{R}$					$\lvert g(z)\rvert > m$
as	if $\forall$		$\bullet\ \exists$		$\bullet\ \forall\, z : \Sigma \mid$	$\bullet$	
$z \longrightarrow \alpha$				$\delta : \mathbb{R}_+$	$0 < \lvert z - \alpha\rvert < \delta$		
$z \longrightarrow \infty$				$M : \mathbb{R}$	$\lvert z\rvert \geqslant M$		